Stability of Operators and Operator Semigroups

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ISBN 978 3 0346 0194 8
Format (B x L): 15,5 x 23,5 cm
Gewicht: 548 g
Evolving systems can be modelled using a discrete or a continuous time scale. The discrete model leads to a map $\varphi$ and its powers $\varphi^n$ on some state space $\Omega$, while the continuous model is given by a (semi)flow $(\varphi_t)_{t \geq 0}$ on $\Omega$ satisfying $\varphi_0 = Id$ and $\varphi_{t+s} = \varphi_t \varphi_s$. In many situations, the state space $\Omega$ is a Banach space and the maps are linear and bounded. In this case, we will use the notations $T$ and $T(t)$ instead of $\varphi$ and $\varphi_t$.

In this book, our focus is on the various concepts of stability of such linear systems, where by stability we mean convergence to zero of $\{T^n\}_{n=1}^\infty$ or $(T(t))_{t \geq 0}$ in a sense to be specified. This property is crucial for study of the qualitative behaviour of dynamical systems, even in the non-linear case. We adopt the general philosophy of a parallel treatment of both the discrete and the continuous case, systematically comparing methods and results. We try to give a reasonably complete picture mentioning (most of) the relevant results, a strategy that has, by the way, helped us to identify a number of natural open problems. However, we do not discuss the case of positive operators and semigroups on Banach lattices, referring the reader to, e.g., Nagel (ed.) [196] or the recent monograph of Emel’yanov [75] for this topic. We also tried to minimise overlap with the monographs of van Neerven [204] on asymptotics in the continuous case and Müller [191] in the discrete case. Instead we emphasise the connections of stability in operator theory to its analogues in ergodic theory and harmonic analysis.

In the following we summarise the content of the book.

Chapter I gives an overview on some functional analytic tools needed later. Besides the classical decomposition theorems of Jacobs–Glicksberg–de Leeuw for compact semigroups, we recall the powerful notion of the cogenerator of a $C_0$-semigroup. Finally, we present one of our main concepts for the investigation of stability of $C_0$-semigroups, the Laplace inversion formula, which can also be seen as an extension of the Dunford functional calculus for exponential functions.
In Chapter II we investigate the behaviour of the powers of a bounded linear operator on a Banach space. As a first step, we study power boundedness. Here and later, behaviour of the resolvent of the operator near the unit circle plays a crucial role. In particular, we give a resolvent characterisation for power boundedness on Hilbert spaces using the $L^2$-norm of the resolvent on circles with radius greater than 1. We further study the related notion of polynomial boundedness which is, surprisingly, easier to characterise than power boundedness.

Stability is the topic in the rest of the chapter. While uniform stability is characterised by the spectral radius, we discuss strong stability and give various characterisations of it, both classical and recent. When turning to weak stability, we encounter a phenomenon, well-known in ergodic theory and described by Katok and Hasselblatt [143, p. 748] as follows\(^{1}\)

"... It [mixing] is, however, one of those notions, that is easy and natural to define but very difficult to study..."

Thus very few results concerning weak stability are known, a sufficient condition in terms of the resolvent being one of the exceptions.

We then introduce the concept of almost weak stability and give various equivalent conditions. From this we see that almost weak stability, while looking artificial, is much easier to characterise than weak stability. In particular, if the operator has relatively compact orbits, then almost weak stability is equivalent to "no point spectrum on the unit circle".

Almost weak stability turns out to be fundamentally different from weak stability, as we see in Section 5. We show that a "typical" (in the sense of Baire) unitary operator, a "typical" isometry, and a "typical" contraction on a separable Hilbert space are almost weakly, but not weakly, stable. This is an operator theoretic counterpart to the classical theorems of Halmos [116] and Rohlin [221] on weakly and strongly mixing transformations on a probability space.

We close the chapter by characterising stability and power boundedness of operators on Hilbert spaces via Lyapunov equations.

In Chapter III we turn our attention to the time continuous case and consider $C_0$-semigroups $(T(t))_{t \geq 0}$ on Banach spaces. As in the previous chapter we discuss boundedness and stability of $(T(t))_{t \geq 0}$ and try to characterise these properties by the generator, its spectrum and resolvent. We start with boundedness and discuss in particular the quite recent characterisation of a bounded $C_0$-semigroup on a Hilbert space by Gomilko [104] and Shi, Feng [231]. Their integrability condition for the resolvent on vertical lines (Theorem 1.11) is the key to the resolvent approach to boundedness and stability, as first introduced by van Casteren [45, 46]. We then characterise generators of polynomially bounded semigroups (i.e., semigroups growing not faster than a polynomial) in terms of various resolvent conditions.

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\(^{1}\)Mixing corresponds to weak stability.
Introduction

A discussion of uniform exponential stability of \( C_0 \)-semigroups follows, a notion which is more difficult to characterise than its discrete analogue. Besides Gearhart’s generalisation to Hilbert spaces of Liapunov’s stability theorem and the Datko–Pazy theorem, we extend Gearhart’s theorem to Banach spaces.

Strong stability is the subject of the following section containing the classical results of Sz.-Nagy and Foiaş, Lax and Phillips and the more recent theorem by Arendt, Batty [9], Lyubich and Vû [180]. We then discuss the resolvent approach developed by Tomilov [243].

Weak stability and its characterisations, partly classical and partly quite recent, is the next topic. We emphasise that, as in the discrete case, weak stability is much less understood than its strong and uniform analogues, still leaving many open questions. For example, it is not clear how to characterise weak stability in terms of the resolvent of the generator.

Next, we look at almost weak stability (Definition 5.3), which is closely related to weak stability but occurs much more frequently. We give various equivalent conditions, analogous to the discrete case, and present a concrete example of an almost weakly, but not weakly, stable semigroup. We finally present category theorems analogous to the discrete ones stating that a “typical” (in the sense of Baire) unitary \( C_0 \)-group as well as a “typical” isometric \( C_0 \)-semigroup on a separable infinite-dimensional Hilbert space are almost weakly, but not weakly, stable.

Characterisation of stability and boundedness of \( C_0 \)-semigroups on Hilbert spaces via Lyapunov equations is the subject of the last section.

Chapter IV relates our stability concepts to weakly and strongly mixing transformations and flows in ergodic theory and (via the spectral theorem) to Rajchman and non-Rajchman measures in harmonic analysis. In addition, “typical” behaviour of contractive operators and semigroups on Hilbert spaces is studied using the notion of rigidity, extending the corresponding results in the previous two chapters.

In Chapter V we build bridges between discrete systems \( \{T^n\}_{n=0}^{\infty} \) and continuous systems \( \{T(t)\}_{t \geq 0} \). We start by embedding a discrete system into a continuous one. More precisely, we ask for which operators \( T \) there exists a \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) with \( T = T(1) \). We discuss some spectral conditions for embeddability and give classes of examples for which such a \( C_0 \)-semigroup does or does not exist. The general embedding question is still open as well as its analogues in ergodic and in measure theory. In Section 2 we discuss the connection between a \( C_0 \)-semigroup \( T(\cdot) \) and its cogenerator \( V \) and its induced discrete system \( \{V^n\}_{n=0}^{\infty} \). On Hilbert spaces there are great similarities between asymptotic properties of the two systems. We give classical and more recent results, discuss difficulties, examples, and open questions.

Acknowledgment. My deepest thanks go to Rainer Nagel, whose strong encouragement and support, deep mathematical insight and permanent optimism has been an invaluable help.
Introduction

I am very grateful to Wolfgang Arendt, Ralph Chill, Bálint Farkas, Jerome Goldstein, Sen-Zhong Huang, Dávid Kunszenti-Kovács, Yuri Latushkin, Mariusz Lemańczyk, Jan van Neerven, Werner Ricker, Yuri Tomilov, Ulf Schlotterbeck, András Serény, Jaroslav Zémanek and Hans Zwart for helpful and interesting discussions and comments on this book.

The cordial working atmosphere in the AGFA group (Arbeitsgemeinschaft Funktionalanalysis) in Tübingen has been very stimulating and helpful. Special thanks go to András Batkai, Fatih Bayazit, Ulrich Groh, Retha Heymann, Bernd Klöss, Dávid Kunszenti-Kovács, Felix Pogorzelski, Ulf Schlotterbeck, Marco Schreiber, and Pavel Zorin for reading preliminary versions of the manuscript.

Support by the Dr. Meyer-Struckmann Stiftung (via the Studienstiftung des deutschen Volkes), the European Social Fund and the Ministry Of Science, Research and the Arts Baden-Württemberg is gratefully acknowledged.

I deeply thank my parents for their hearty encouragement and permanent support, as well as Anna M. Vishnyakova for teaching me how to become a mathematician.