Quantum field theory in curved spacetime has been remarkably fruitful. It can be used to explain how the large-scale structure of the universe and the anisotropies of the cosmic background radiation that we observe today first arose. Similarly, it provides a deep connection between general relativity, thermodynamics, and quantum field theory. This book develops quantum field theory in curved spacetime in a pedagogical style, suitable for graduate students.

The authors present detailed, physically motivated derivations of cosmological and black hole processes in which curved spacetime plays a key role. They explain how such processes in the rapidly expanding early universe leave observable consequences today, and how, in the context of evaporating black holes, these processes uncover deep connections between gravitation and elementary particles. The authors also lucidly describe many other aspects of free and interacting quantized fields in curved spacetime.

Leonard E. Parker is Distinguished Professor Emeritus in the Physics Department at the University of Wisconsin, Milwaukee. In the 1960s, he was the first to use quantum field theory to show that the gravitational field of the expanding universe creates elementary particles from the vacuum.

David J. Toms is Reader in Mathematical Physics in the School of Mathematics and Statistics at Newcastle University. His research interests include the formalism of quantum field theory and its applications, and his most recent interests involve the use of quantum field theory methods in low energy quantum gravity.
R. Penrose and W. Rindler Spinors and Space-Time Volume 2: Spinor and Twistor Methods in Space-Time Geometry†
S. Pokorski Gauge Field Theories, 2nd edition†
J. Polchinski String Theory Volume 1: An Introduction to the Bosonic String
J. Polchinski String Theory Volume 2: Superstring Theory and Beyond
V. N. Popov Functional Integrals and Collective Excitations†
R. J. Rivers Path Integral Methods in Quantum Field Theory†
R. G. Roberts The Structure of the Proton: Deep Inelastic Scattering†
C. Rovelli Quantum Gravity†
W. C. Saslaw Gravitational Physics of Stellar and Galactic Systems†
H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt Exact Solutions of Einstein’s Field Equations, 2nd edition†
J. Stewart Advanced General Relativity†
T. Thiemann Modern Canonical Quantum General Relativity
D. J. Toms The Schwinger Action Principle and Effective Action
A. Vilenkin and E. P. S. Shellard Cosmic Strings and Other Topological Defects†
R. S. Ward and R. O. Wells, Jr Twistor Geometry and Field Theory†
J. R. Wilson and G. J. Mathews Relativistic Numerical Hydrodynamics

† Issued as a paperback
Quantum Field Theory in Curved Spacetime
Quantized Fields and Gravity

LEONARD E. PARKER
University of Wisconsin, Milwaukee

DAVID J. TOMS
University of Newcastle upon Tyne
## Contents

**Preface**  
page xi  

**Acknowledgments**  
xiii  

**Conventions and notation**  
xxv  

1 Quantum fields in Minkowski spacetime  
1.1 Canonical formulation  
2  
1.2 Particles  
13  
1.3 Vacuum energy  
21  
1.4 Charged scalar field  
24  
1.5 Dirac field  
27  
1.6 Angular momentum and spin  
32  

2 Basics of quantum fields in curved spacetimes  
2.1 Canonical quantization and conservation laws  
37  
2.2 Scalar field  
43  
2.3 Cosmological model: Arbitrary asymptotically static time dependence  
47  
2.4 Particle creation in a dynamic universe  
51  
2.5 Statistics from dynamics: Spin-0  
54  
2.6 Conformally invariant non-interacting field  
56  
2.7 Probability distribution of created particles  
58  
2.8 Exact solution with particle creation  
61  
2.9 High-frequency blackbody distribution  
63  
2.10 de Sitter spacetime  
64  
2.11 Quantum fluctuations and early inflation  
73  
2.12 Quantizing the inflaton field perturbations  
77  
2.13 A word on interacting quantized fields and on algebraic quantum field theory in curved spacetime  
88  
2.14 Accelerated detector in Minkowski spacetime  
91  

3 Expectation values quadratic in fields  
3.1 Adiabatic subtraction and physical quantities  
93  
3.2 Energy-momentum tensor from trace anomaly  
107  
3.3 Renormalization in general spacetimes  
110  
3.4 Gaussian approximation to propagator  
115
## Contents

3.5 Approximate energy-momentum tensor in Schwarzschild, de Sitter, and other static Einstein spacetimes 118  
3.6 $R$-summed form of propagator 129  
3.7 $R$-summed action and cosmic acceleration 131  
3.8 Normal coordinate momentum space 134  
3.9 Chiral current anomaly caused by spacetime curvature 144  

4 Particle creation by black holes 152  
4.1 Introduction 152  
4.2 Classical considerations 153  
4.3 Quantum aspects 162  
4.4 Energy-momentum tensor with Hawking flux 174  
4.5 Back reaction to black hole evaporation 178  
4.6 Trans-Planckian physics in Hawking radiation and cosmology 180  
4.7 Further topics: Closed timelike curves; closed-time-path integral 182  

5 The one-loop effective action 184  
5.1 Introduction 184  
5.2 Preliminary definition of the effective action 185  
5.3 Regularization of the effective action 190  
5.4 Effective action for scalar fields: Some examples 200  
5.5 The conformal anomaly and the functional integral 216  
5.6 Spinors in curved spacetime 221  
5.7 The effective action for spinor fields 245  
5.8 Application of the effective action for spinor fields 253  
5.9 The axial, or chiral, anomaly 257  

6 The effective action: Non-gauge theories 268  
6.1 Introduction 268  
6.2 The Schwinger action principle 271  
6.3 The Feynman path integral 277  
6.4 The effective action 280  
6.5 The geometrical effective action 283  
6.6 Perturbative expansion of the effective action 295  
6.7 Renormalization of an interacting scalar field theory 302  
6.8 The renormalization group and the effective action 318  
6.9 The effective potential 322  
6.10 The renormalization of the non-linear sigma model 331  
6.11 Formal properties of the effective action 337  

Appendix 344
## Contents

7 The effective action: Gauge theories 348
  7.1 Introduction 348
  7.2 Gauge transformations 350
  7.3 The orbit space and the gauge condition 359
  7.4 Field space reparameterization and the Killing equation 368
  7.5 The connection and its consequences 375
  7.6 The functional measure for gauge theories 385
  7.7 Gauge-invariant effective action 390
  7.8 Yang–Mills theory, concluded 399
  7.9 Scalar quantum electrodynamics 407
    Appendix 420

Appendix: Quantized Inflaton Perturbations 422

References 426
Index 445
Preface

The success of Einstein’s theory of general relativity convincingly demonstrates that the classical gravitational field is a manifestation of the curvature of spacetime. Similarly, quantum field theory in Minkowski spacetime successfully describes the behavior of elementary particles over a wide range of energies. It has proved notoriously difficult to understand how gravity fits with the quantum attribute of the fields that transmit the other forces of nature. Leading attempts to combine gravitation and quantum field theory include string theory and loop quantum gravity. String theory attempts to describe elementary particles, including the graviton, as quantized excitations of systems of strings and D-branes in a higher-dimensional space. Loop quantum gravity attempts to describe the structure of spacetime itself in terms of quantized loops. At energies much below the Planck scale, these theories reduce to descriptions of quantized fields propagating in a curved spacetime having a metric described by Einstein’s gravitational field equations with additional higher-order curvature corrections.

Quantum field theory in curved spacetime is the framework for describing elementary particles and gravitation at energies below the Planck scale. This theory has had striking successes. It has shown how gravitation and quantum field theory are intimately connected to give a consistent description of black holes having entropy and satisfying the second law of thermodynamics; and it has shown how the inhomogeneities and anisotropies we observe today in the cosmic microwave background and in the large-scale structure of the universe were created in a brief stage of very rapid expansion of the universe, known as inflation.

This book should give the reader a deep understanding of the principles of quantum field theory in curved spacetime and of their applications to the early universe, renormalization, black holes, and effective action methods for interacting fields in curved spacetime, including gauge fields. It is aimed at graduate students and researchers and would be appropriate as the basis for a graduate course. We have tried to be pedagogical in our presentation.

If the students have had an introduction to quantum field theory in Minkowski spacetime, then Chapter 1 could be skipped and returned to only when particular topics are unfamiliar. In that case, the instructor can expect to finish Chapters 2 through 5 in a one-semester course, depending on how much detail is covered. Particle creation by the expansion of the universe and by black holes would be covered in such a course. In a two-semester course, the instructor can expect to
go through the whole book, including renormalization of interacting fields, the renormalization group, and the effective action for Yang–Mills fields in curved spacetime.

We expect the reader to have some understanding of general relativity, at least at the introductory level. The books by Hartle (2003), Misner et al. (1973), Wald (1984), or Weinberg (1972) provide more than sufficient background on general relativity. For additional related material on quantum field theory in curved spacetime, we recommend that the reader consult the outstandingly comprehensive treatment of the subject by Birrell and Davies (1982), and the books by Fulling (1989) and Wald (1994).
Acknowledgments

Leonard E. Parker is grateful to the late Sidney Coleman. He supervised my PhD thesis at Harvard from 1962 through 1965, in which I used quantum field theory to discover and thoroughly investigate the creation of elementary particles by the gravitational field of the expanding universe, including the role of conformal invariance and other aspects of this fundamentally important process. I am grateful as well to Steven A. Fulling, Lawrence H. Ford, Timothy S. Bunch, and my coauthor David J. Toms, who were my postdoctoral associates from 1973 through the early 1980s. Their work helped push the boundaries of quantum field theory in curved spacetime into new and fruitful territory. It is with pleasure that I thank as well my other postdoctoral associates, including the late Chaim Charach, Ian Jack, Atshushi Higuchi, Jonathan Z. Simon, Jorma Louko, Yoav Peleg, Alpan Raval, Daniel A. T. Vanzella, and Gonzalo J. Olmo. It was a privilege to work with these truly outstanding researchers. Among my other collaborators on topics involving quantum field theory in curved spacetime, I give special thanks to Paul Anderson, Jacob D. Bekenstein, Robert Caldwell, Bei-Lok Hu, and José Navarro-Salas. The PhD students who worked with me on topics related to quantum field theory in curved spacetime are Prakash Panangaden, Luis O. Pimentel, Todd K. Leen, Esteban Calzetta, Yang Zhang, Sukanta Bose, Gerald Graef, William Komp, and Laura Mersini. Matthew Glenz, one of my current PhD students, carefully read chapters 1 through 4 and pointed out many misprints and unclear passages. I am grateful to my former postdoctoral associate Alpan Raval for helping with the writing of Sections 3.5, 4.4, and 4.5. I thank the US National Science Foundation for supporting my research on quantized fields, gravitation, and cosmology for more than 35 years. This support was of great help. Above all, I thank my wife, Gloria, and children, David, Michael, and Deborah, for their support and encouragement during the writing of this book.

David Toms would like, first of all, to express his gratitude to R. C. Roeder, who first suggested to him that quantum field theory in curved spacetime would make a good topic for a PhD thesis, and to P. J. O’Donnell who supervised his PhD studies and gave him free rein to follow his interests, along with some good advice. While a postgraduate student at the University of Toronto I benefitted greatly from discussions with C. C. Dyer and E. Honig. The Natural Sciences and Engineering Research Council of Canada sponsored my first postdoctoral fellowship at Imperial College, London. While there I profited from interactions with M. J. Duff, L. H. Ford, C. J. Isham, G. Kunstatter, and M. Pilati. My first
encounter with my coauthor, Leonard E. Parker, was as a postdoctoral associate at the University of Wisconsin at Milwaukee where we established a fruitful collaboration that led eventually to this book. I am also grateful to J. Friedman and D. M. Witt for valuable discussions while I was in Milwaukee. My colleagues at Newcastle deserve thanks, especially P. C. W. Davies and I. G. Moss. I would like to acknowledge my collaborators not previously mentioned, J. Balakrishnan, K. Kirsten, and H. P. Leivo, for their assistance. I have had a number of postgraduate students who worked with me on aspects of quantum field theory among whom are M. Burgess, E. J. Copeland, P. Ellicott, A. Flachi, S. R. Huggins, G. Huish, and I. Russell. P. McKay pointed out some misprints in Chapter 7. While at Newcastle I have received funding from the Sciences and Engineering Research Council and the Nuffield Foundation. My wife Linda was very understanding and supportive during the many hours taken to write this book – I express my love and gratitude for this.

Both authors are extremely grateful to Bei-Lok Hu for his advice through the years, and his comments on an early version of the manuscript.
Conventions and notation

We have tried to maintain consistency between our book and that of Birrell and Davies (1982) wherever possible. Our sign conventions are \((-\ldots-\)\) in the notation of Misner et al. (1973). More explicitly, an outline of our basic notation is the following:

- \(\Re\) and \(\Im\) denote the real and imaginary parts of any expression;
- \(\text{div}\) denotes the divergent part (pole part if dimensional regularization is used);
- we use units with \(c = \hbar = 1\), and often \(G = 1\);
- spacetime dimension is \(n\) in general, often with \(n = 4\);
- Minkowski metric: \(\eta_{\mu\nu}\) is diagonal with eigenvalues \((+1, -1, \ldots, -1)\);
- ordinary partial derivative of \(\psi\) denoted by \(\partial_\mu \psi\) or \(\psi_\mu\);
- curved spacetime metric: \(g_{\mu\nu}\) with inverse metric \(g^{\mu\nu}\);
- invariant volume element: \(dv = |\det(g_{\mu\nu})|^{1/2} d^n x\);
- Christoffel connection: 
  \[
  \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} = \frac{1}{2} g^{\lambda\sigma}(g_{\sigma\nu,\mu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma})
  \]
- covariant derivative of \(\psi\) denoted by \(\nabla_\mu \psi\) or \(\psi_\mu\);
- d’Alembertian, or wave, operator: \(\Box = \nabla\mu \nabla_\mu\);
- Riemann tensor: 
  \[
  R^\lambda_{\tau\mu\nu} = \Gamma^\lambda_{\tau\mu,\nu} - \Gamma^\lambda_{\tau\nu,\mu} + \Gamma^\lambda_{\nu\sigma} \Gamma^\sigma_{\tau\mu} - \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\tau\nu}
  \]
- Ricci tensor: 
  \[
  R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}
  \]
- Dirac matrices in flat spacetime follow Bjorken and Drell (1964). (See Chapter 5 for a complete discussion.)

Other notation is introduced as needed.