STOCHASTIC CALCULUS AND DIFFERENTIAL EQUATIONS FOR PHYSICS AND FINANCE

Stochastic calculus provides a powerful description of a specific class of stochastic processes in physics and finance. However, many econophysicists struggle to understand it. This book presents the subject simply and systematically, giving graduate students and practitioners a better understanding and enabling them to apply the methods in practice.

The book develops Itô calculus and Fokker–Planck equations as parallel approaches to stochastic processes, using those methods in a unified way. The focus is on nonstationary processes, and statistical ensembles are emphasized in time series analysis. Stochastic calculus is developed using general martingales. Scaling and fat tails are presented via diffusive models. Fractional Brownian motion is thoroughly analyzed and contrasted with Itô processes. The Chapman–Kolmogorov and Fokker–Planck equations are shown in theory and by example to be more general than a Markov process. The book also presents new ideas in financial economics and a critical survey of econometrics.

JOSEPH L. McCauley is Professor of Physics at the University of Houston. During his career he has contributed to several fields, including statistical physics, superfluids, nonlinear dynamics, cosmology, econophysics, economics, and finance theory.
STOCHASTIC CALCULUS AND DIFFERENTIAL EQUATIONS FOR
PHYSICS AND FINANCE

JOSEPH L. McCauley
University of Houston
For our youngest ones,
Will, Justin, Joshua, Kayleigh, and Charlie
# Contents

**Abbreviations**

<table>
<thead>
<tr>
<th>Introduction</th>
<th>page xi</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Random variables and probability distributions</td>
<td>5</td>
</tr>
<tr>
<td>1.1 Particle descriptions of partial differential equations</td>
<td>5</td>
</tr>
<tr>
<td>1.2 Random variables and stochastic processes</td>
<td>7</td>
</tr>
<tr>
<td>1.3 The n-point probability distributions</td>
<td>9</td>
</tr>
<tr>
<td>1.4 Simple averages and scaling</td>
<td>10</td>
</tr>
<tr>
<td>1.5 Pair correlations and 2-point densities</td>
<td>11</td>
</tr>
<tr>
<td>1.6 Conditional probability densities</td>
<td>12</td>
</tr>
<tr>
<td>1.7 Statistical ensembles and time series</td>
<td>13</td>
</tr>
<tr>
<td>1.8 When are pair correlations enough to identify a stochastic process?</td>
<td>16</td>
</tr>
<tr>
<td>Exercises</td>
<td>17</td>
</tr>
</tbody>
</table>

| 2 Martingales, Markov, and nonstationarity | 18 |
| 2.1 Statistically independent increments | 18 |
| 2.2 Stationary increments | 19 |
| 2.3 Martingales | 20 |
| 2.4 Nonstationary increment processes | 21 |
| 2.5 Markov processes | 22 |
| 2.6 Drift plus noise | 22 |
| 2.7 Gaussian processes | 23 |
| 2.8 Stationary vs. nonstationary processes | 24 |
| Exercises | 26 |

| 3 Stochastic calculus | 28 |
| 3.1 The Wiener process | 28 |
| 3.2 Ito’s theorem | 29 |
3.3 Ito’s lemma 30
3.4 Martingales for greenhorns 31
3.5 First-passage times 33
   Exercises 35

4 Ito processes and Fokker–Planck equations 37
4.1 Stochastic differential equations 37
4.2 Ito’s lemma 39
4.3 The Fokker–Planck pde 39
4.4 The Chapman–Kolmogorov equation 41
4.5 Calculating averages 42
4.6 Statistical equilibrium 43
4.7 An ergodic stationary process 45
4.8 Early models in statistical physics and finance 45
4.9 Nonstationary increments revisited 48
   Exercises 48

5 Selfsimilar Ito processes 50
5.1 Selfsimilar stochastic processes 50
5.2 Scaling in diffusion 51
5.3 Superficially nonlinear diffusion 53
5.4 Is there an approach to scaling? 54
5.5 Multiaffine scaling 55
   Exercises 56

6 Fractional Brownian motion 57
6.1 Introduction 57
6.2 Fractional Brownian motion 57
6.3 The distribution of fractional Brownian motion 60
6.4 Infinite memory processes 61
6.5 The minimal description of dynamics 62
6.6 Pair correlations cannot scale 63
6.7 Semimartingales 64
   Exercises 65

7 Kolmogorov’s pdes and Chapman–Kolmogorov 66
7.1 The meaning of Kolmogorov’s first pde 66
7.2 An example of backward-time diffusion 68
7.3 Deriving the Chapman–Kolmogorov equation for an Ito process 68
   Exercise 70
Contents

8 Non-Markov Ito processes 71
  8.1 Finite memory Ito processes? 71
  8.2 A Gaussian Ito process with 1-state memory 72
  8.3 McKean’s examples 74
  8.4 The Chapman–Kolmogorov equation 78
  8.5 Interacting system with a phase transition 79
  8.6 The meaning of the Chapman–Kolmogorov equation 81
    Exercise 82
9 Black–Scholes, martingales, and Feynman–Kac 83
  9.1 Local approximation to sdes 83
  9.2 Transition densities via functional integrals 83
  9.3 Black–Scholes-type pdes 84
    Exercise 85
10 Stochastic calculus with martingales 86
  10.1 Introduction 86
  10.2 Integration by parts 87
  10.3 An exponential martingale 88
  10.4 Girsanov’s theorem 89
  10.5 An application of Girsanov’s theorem 91
  10.6 Topological inequivalence of martingales with Wiener processes 93
  10.7 Solving diffusive pdes by running an Ito process 96
  10.8 First-passage times 97
  10.9 Martingales generally seen 102
    Exercises 105
11 Statistical physics and finance: A brief history of each 106
  11.1 Statistical physics 106
  11.2 Finance theory 110
    Exercise 115
12 Introduction to new financial economics 117
  12.1 Excess demand dynamics 117
  12.2 Adam Smith’s unreliable hand 118
  12.3 Efficient markets and martingales 120
  12.4 Equilibrium markets are inefficient 123
  12.5 Hypothetical FX stability under a gold standard 126
  12.6 Value 131
Contents

12.7 Liquidity, reversible trading, and fat tails vs. crashes 132
12.8 Spurious stylized facts 143
12.9 An sde for increments 146
   Exercises 147
13 Statistical ensembles and time-series analysis 148
   13.1 Detrending economic variables 148
   13.2 Ensemble averages and time series 149
   13.3 Time-series analysis 152
   13.4 Deducing dynamics from time series 162
   13.5 Volatility measures 167
      Exercises 168
14 Econometrics 169
   14.1 Introduction 169
   14.2 Socially constructed statistical equilibrium 172
   14.3 Rational expectations 175
   14.4 Monetary policy models 177
   14.5 The monetarist argument against government intervention 179
   14.6 Rational expectations in a real, nonstationary market 180
   14.7 Volatility, ARCH, and GARCH 192
      Exercises 195
15 Semimartingales 196
   15.1 Introduction 196
   15.2 Filtrations 197
   15.3 Adapted processes 197
   15.4 Martingales 198
   15.5 Semimartingales 198
      Exercise 199

References 200
Index 204
Abbreviations

B(t), Wiener process
x(t) or X(t), random variable at time t in a stochastic process
f_n(x_n, t_n; ...; x_1, t_1), n-point density of a continuous random variable x at n different
times t_1 ≤ t_2 ≤ ... ≤ t_n.

p_2(x, t|y, s), conditional density to get x at time t, given that y was observed at
time s < t.

⟨x(t)⟩_c = \int dx p_2(x, t|y, s), avg. of x at time t conditioned on having observed y
at time s. Using a bracket to denote an average is standard in physics since the
time of Dirac.

A(x, t), dynamical variable, meaning a function of a random variable x and also
the time t.

⟨A(t)⟩ = ∫ dx A(x, t) f_1(x, t), absolute average of a dynamical variable A.

⟨x(t)y(s)⟩ = ∫ dx dy x f_2(x, y), pair correlation function

⟨x(t)⟩ = ∫ dx xp_2(x, t|y, s) f_1(y, s), absolute average of x at time t; ⟨x(t)⟩ =
∫ dx A(x) f_1(x, t) since ∫ dy p_2(x, t|y, s) = 1.

⟨x(t)⟩ = ∫ dx xp_2(x, t|y, s) = y, martingale process
x(t, T) = x(t + T) − x(t), an increment/displacement/difference

⟨x^2(t, T)⟩, mean square fluctuation about an arbitrary point x observed at time t.

dX = R(X, t) dt + b(X, s) dB(t), Ito process;

b^2(x, t) = D(x, t) is the diffusion coefficient

M(t), a martingale in Ito calculus, dM(t) = ±√D(M, t) dB(t)

{X} = ∫ dX^2 where (dX)^2 = D(X, t) dt^1

{X, Y} = \frac{1}{2} (\{X + Y\} − \{X − Y\})

fBm, fractional Brownian motion, a mathematical model with stationary incre-
ments and long-time correlations

ratex, rational expectations, a mathematized ideology

1 This is a special notation used in Chapter 10 where stochastic calculus is extended to martingales dX =
b(X, t) dB(t). It differs from Durrett’s notation because we use his bracket symbol ⟨⟩ to denote averages.