Gravitational $N$-Body Simulations

This book discusses in detail all the relevant numerical methods for the classical $N$-body problem. It demonstrates how to develop clear and elegant algorithms for models of gravitational systems and explains the fundamental mathematical tools needed to describe the dynamics of a large number of mutually attractive particles. Particular attention is given to the techniques needed to model astrophysical phenomena such as close encounters and the dynamics of black-hole binaries. The author reviews relevant work in the field and covers applications to the problems of planetary formation and star-cluster dynamics, both of Pleiades-type and globular clusters.

Self-contained and pedagogical, this book is suitable for graduate students and researchers in theoretical physics, astronomy and cosmology.

SVERRE AARSETH received his B.Sc. from the University of Oslo in 1959 and his Ph.D. from the University of Cambridge in 1963. After a few years as research assistant to Professor F. Hoyle, he joined the newly created Institute of Theoretical Astronomy in 1967 (which then became the Institute of Astronomy in 1972). His entire career has been spent at this Institute as a post-doctoral research fellow, giving him complete freedom to devote himself exclusively to all aspects of the modern $N$-body problem. The stimulating Cambridge environment has been ideal for establishing collaborations with visiting astronomers. Dr Aarseth has developed a unique set of codes that include the latest techniques, and are now publicly available. These codes are suitable for laptops and workstations as well as for the most powerful special-purpose computers.
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†Issued as a paperback
Gravitational $N$-Body Simulations

SVERRE J. AARSETH
Institute of Astronomy
University of Cambridge
To the world’s wild and magical places
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Preface

This book spans my entire life as a research worker at Cambridge. The circumstances that created this opportunity were based entirely on luck and this aspect played a vital part during subsequent developments. In the following chapters, I have tried to give details of the most relevant methods used in so-called ‘direct integration’ of the classical N-body problem, a method of attack somewhat analogous to scaling a mountain the hard way. This has been enhanced by an extensive discussion of the main algorithms implemented in the associated computer codes. A comprehensive review of related work in the field over the last 40 years is also presented. Throughout the term N-body simulations is used exclusively for methods based on direct summation, in keeping with tradition.

Although a wide range of problems is covered, the emphasis is on the dynamics of star clusters. This involves many aspects of stellar evolution. It is fortuitous that the University of Cambridge has a long tradition in this field that dates back to Eddington and Jeans. Fred Hoyle continued this school, which eventually gave rise to the application of synthetic stellar evolution. This subject was pioneered entirely at the Institute, mainly by the sequential efforts of Peter Eggleton, Christopher Tout and Jarrod Hurley, whose work has been vital for realistic star cluster simulations.

I would like to acknowledge the assistance of colleagues who read and commented critically on various chapters – Raul de la Fuente Marcos, Douglas Heggie, Jarrod Hurley, Pavel Kroupa, Derek Richardson, Rainer Spurzem, Christopher Tout and Mark Wilkinson. Specific suggestions for improvements of the contents were made by Doug Lin, Rosemary Mardling and HongSheng Zhao. My thanks also go to Robert Izzard who did most of the figures.

Among my many collaborators, I am especially indebted to Avishai Dekel, Richard Gott, Douglas Heggie, Jarrod Hurley, Pavel Kroupa, Mike Lecar, Doug Lin, Jun Makino, Rosemary Mardling, Steve McMillan,
Preface

Seppo Mikkola, Rainer Spurzem, Christopher Tout and Khalil Zare. Likewise, the pioneers Michel Hénon, Sebastian von Hoerner and Roland Wielen provided impetus and advice in the early days. More recently, Piet Hut has acted as a catalyst for stimulating new developments.

Claude Froeschlé, Douglas Heggie, E.L. Stiefel, Victor Szebenely and Khalil Zare educated and influenced me in the fundamental topic of regularization. Moreover, the contributions of Seppo Mikkola to our collaborations in this subject over the past 15 years have been invaluable.

Lastly, in the scientific field, I have benefited greatly from the technical assistance given to me by Jun Makino, Steve McMillan and Rainer Spurzem. My sincere thanks are due to Jun Makino and Makoto Taiji who designed the special-purpose HARP-2 computer that occupied my office from 1994, and likewise to Jun Makino who is the driving force behind GRAPE-6, which has recently become the simulator’s dream machine.

I made an auspicious start at the newly created Institute of Theoretical Astronomy, founded in 1967 by Sir Fred Hoyle. He was also my Ph.D. supervisor and directly responsible for independently suggesting $N$-body simulations as a research topic. For all this I am immensely grateful.

My subsequent career would not have been possible without strong support from the Directors of the Institute of Astronomy since the name change in 1972, Donald Lynden-Bell, Sir Martin Rees, Richard Ellis and Douglas Gough. They allowed me complete freedom to pursue my singular interest in dynamics. I have also depended utterly on continuous post-doctoral funding since 1963 by the Government Research Establishments that have undergone several name changes but lately are known as PPARC.

On the personal side, I would like to express my deepest thanks to Patricia who supported my work and also endured my other obsessions of chess and mountaineering. I am very grateful to my mother and father for their help and encouragement during the difficult formative years in Norway. Most of this book was written at the family mountain retreat near beautiful Lake Reinunga, which provided tranquility and inspiration.

Because of my involvement since the beginning, I have taken the opportunity to review the whole subject of $N$-body simulations as defined above. In view of the increasing activity this is a daunting task, particularly when it comes to making critical comments. In such circumstances my opinion is often expressed instead of merely quoting published work. I apologize for significant omissions and take full responsibility for any misrepresentations that are bound to occur. This book has been in preparation for a very long time. I would like to thank my editor, Tamsin van Essen, and the staff at Cambridge University Press for their patience and advice. Special thanks are due to my copy editor, Robert Whitelock, for his critical appraisal.
In conclusion, our field has undergone a remarkable development, fuelled by an exponential growth of computing power as well as software advances. Although the beginnings were modest and the developments slow, it has now blossomed into a fully fledged scientific activity. For the longer term, further progress is only possible if we attract the younger generation to seek new challenges and enrich our subject. It is therefore my hope that this book will prove timely and serve a practical purpose. Finally, the dedication reflects my many sources of inspiration, whether it be the awesome beauty of the Atacama Desert or more accessible wildlife environments. May our planet’s fragile ecosystem and rich diversity be preserved for future enjoyment.

Sverre Aarseth

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