Chapter I

Asymptotic Behavior of Generalized Functions

0 Preliminaries

0.1. We denote by $\mathbb{R}$ and $\mathbb{N}$ the sets of real and natural numbers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The following notation will be used. If $x = (x_1, \ldots, x_n), \ y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, then $x \cdot y = x_1 y_1 + \cdots + x_n y_n$; $\|x\|^2 = x_1^2 + \cdots + x_n^2$; $x \geq 0 \iff x_i \geq 0, i = 1, \ldots, n$; $x \to \infty \iff x_i \to \infty, i = 1, \ldots, n$; $\mathbb{R}_+^n = \{x \in \mathbb{R}^n; \ x > 0\}$. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, then $|k| = k_1 + \cdots + k_n, \ k! = k_1! \cdots k_n!$, $x^k = x_1^{k_1} \cdots x_n^{k_n}$; $D^k = \partial^{k_1}/\partial x_1^{k_1} \cdots \partial^{k_n}/\partial x_n^{k_n}$, $f^{(k)} = D^k f$; $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$; $D(a,r)$ denotes the polydisk $\{z \in \mathbb{C}^n; \ |z_i - a_i| < r, \ i = 1, \ldots, n\}$, where $|z_j|^2 = x_j^2 + y_j^2$, $z_j = x_j + iy_j \in \mathbb{C}$. $B(a,r)$ denotes the open ball $\{x \in \mathbb{R}^n, \ \|a - x\| < r\}$ and $H$ is the Heaviside function: $H(t) = 0, t \leq 0$; $H(t) = 1, t > 0$.

0.2. A cone with vertex at zero in $\mathbb{R}^n$ is a non-empty set $\Gamma$ such that $x \in \Gamma$ and $k > 0$ imply $kx \in \Gamma$. The cone $\Gamma$ is called solid if $\text{int} \Gamma \neq \emptyset$. The conjugate cone (dual cone) $\Gamma^*$ to the cone $\Gamma$ is the set $\{\xi \in \mathbb{R}^n; \ x \cdot \xi \geq 0$ for each $x \in \Gamma\}$. It is obvious that $\Gamma^*$ is also a cone which is convex and closed. The cone $\Gamma$ is called acute if $\Gamma^*$ is a solid cone.

0.3. A function $\rho : (a, \infty) \to \mathbb{R}$, $a \in \mathbb{R}_+$, is called regularly varying at infinity [76] if it is positive, measurable, and if there exists a real number $\alpha$ such that for each $x > 0$

$$\lim_{k \to \infty} \frac{\rho(kx)}{\rho(k)} = x^\alpha. \quad (0.1)$$

The number $\alpha$ is called index of regular variation. If $\alpha = 0$, then $\rho$ is called slowly varying at infinity and for such a function the letter “L” will be used. We then have that any regularly varying function can be written as $\rho(x) =$
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$x^\alpha L(x)$, $x > a$, where $L$ is slowly varying. It is known that the convergence of (0.1) is uniform on every fixed compact interval $[a', b')$, $a < a' < b' < \infty$, and that $\rho$ is necessarily bounded (hence integrable) on it [9].

Let $L$ be a slowly varying function at infinity. Then, for each $\varepsilon > 0$,

(i) there exist constants $C_1, C_2 > 0$ and $X > a$ such that

$$C_1 x^{-\varepsilon} \leq L(x) \leq C_2 x^\varepsilon, \quad x \geq X; \quad (0.2)$$

(ii) $\lim_{x \to \infty} x^\varepsilon L(x) = +\infty$, $\lim_{x \to \infty} x^{-\varepsilon} L(x) = 0$.

We know ([9], p. 16) that if $L_2(x) \to \infty$, $x \to \infty$, and $L_1, L_2$ are slowly varying, then $L_1 \circ L_2 = L_1(L_2)$ is slowly varying, as well. Hence, for $x > -\infty$,

$$\lim_{h \to \infty} \frac{L(x + h)}{L(h)} = \lim_{u \to \infty} \frac{L(\log ut)}{L(\log u)} = 1. \quad (0.3)$$

The definition of a regular varying function at zero is similar. For the definition of regularly generalized functions see [156] and [162].

0.4. The class of distributions $f_\alpha$, $\alpha \in \mathbb{R}$, belonging to $S'_+$ (see 0.5.1.) is defined in the following way:

$$f_\alpha(t) = \begin{cases} H(t)t^{\alpha-1}/\Gamma(\alpha), & \alpha > 0, \\ f^{(m)}_{\alpha+m}(t), & \alpha \leq 0, \alpha + m > 0, \end{cases}$$

where $H$ is the Heaviside function, and the derivative $f^{(m)}$ is taken to be in the distributional sense (see [192], Chapter I, §1). We therefore have $f_0 = \delta$, the Dirac delta distribution; $f_m = \delta^{(m)}$, $m \in \mathbb{N}$; and $f_p \ast f_q = f_{p+q}$. We also use the notations $t^\alpha_+ = \Gamma(\alpha + 1)f_{\alpha+1}(t)$ and $t^\alpha_- = (-t)^\alpha_+$, $\alpha \notin \mathbb{N}$.

Let $g \in S'_+$. We denote $g^{(-\alpha)} = f_\alpha \ast g$, $\alpha \in \mathbb{R}$ (* the is convolution symbol).

The sequence $(\delta_m)_{m \in \mathbb{N}} \subset C^\infty(\mathbb{R}^n)$ is called a $\delta$-sequence if:

a) $\text{supp} \, \delta_m \subset [-\alpha_m, \alpha_m]$, $\alpha_m \to 0$, $m \to \infty$; b) $\delta_m \geq 0$, $m \in \mathbb{N}$; c) $\int_{\mathbb{R}^n} \delta_m(t)dt = 1$, $m \in \mathbb{N}$.

If $\varphi \in \mathcal{D}$, then $\delta_m \ast \varphi \to \varphi$, $m \to \infty$ in $\mathcal{D}$, hence $\{\delta_m \ast \varphi; \, m \in \mathbb{N}\}$ is a bounded set in $\mathcal{D}$.

0.5. We will repeat definitions and some basic properties of generalized functions defined as elements of dual spaces.
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0.5.1. The Schwartz spaces of test functions and distributions on $\mathbb{R}^n$ are denoted by $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ and $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$, respectively ($\mathbb{R}^n$ will be omitted wherever $n$ is not fixed). The space $\mathcal{D}$ is a locally convex, barrelled, Montel and complete space. If a filter with a countable basis is weakly convergent in $\mathcal{D}'$, then it is convergent in $\mathcal{D}'$ with the strong topology, as well (cf. [146]).

$\mathcal{S}$ is the space of rapidly decreasing functions and its dual $\mathcal{S}'$ is the space of tempered distributions [146]. For a closed cone $\Gamma \subset \mathbb{R}^n$, $\mathcal{S}'_{\Gamma} = \{ f \in \mathcal{S}'; \text{supp} f \subset \Gamma \}$. In the one-dimensional case $\mathcal{S}'_+ = \{ f \in \mathcal{S}'(\mathbb{R}); \text{supp} f \subset [0, \infty) \}$. Recall ([189]):

$$\mathcal{S}(\Gamma) = \{ \varphi \in C^\infty(\Gamma); \| \varphi \|_p < \infty, p \in \mathbb{N} \},$$

where

$$\| \varphi \|_p = \sup_{x \in \Gamma, |\beta| \leq p} (1 + |x|^2)^{p/2}|\varphi^{(\beta)}(x)|.$$

By $\mathcal{S}_p(\Gamma)$ is denoted the completion of the set $\mathcal{S}(\Gamma)$ with respect to the norm $\| \varphi \|_p$. Note $\mathcal{S}(\Gamma) = \bigcap_{p \in \mathbb{N}_0} \mathcal{S}_p(\Gamma)$ and $\mathcal{S}'(\Gamma) = \bigcup_{p \in \mathbb{N}_0} \mathcal{S}'_p(\Gamma)$, where the intersection and the union have topological meaning. A sequence $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{S}'(\Gamma)$ converges to $f \in \mathcal{S}'(\Gamma)$ if and only if it belongs to some $\mathcal{S}'_p(\Gamma)$ and converges to $f \in \mathcal{S}'_p(\Gamma)$ in the norm of $\mathcal{S}'_p(\Gamma)$. The space $\mathcal{S}'_{\Gamma}$ is isomorphic to $\mathcal{S}'(\Gamma)$ if $\Gamma$ is closed convex solid cone ([189], [192]).

$\mathcal{E}'$ the space of distributions with compact support; it is isomorphic to the dual space of $\mathcal{E} = C^\infty(\mathbb{R}^n)$ (cf. [146]).

$\mathcal{D}_{L^p}, 1 \leq p \leq \infty$, is the space of smooth functions with all derivatives belonging to $L^p$ ([146]), $\mathcal{D}_{L^p} \subset \mathcal{D}_{L^q}$ if $p < q$.

$\mathcal{B}$ is a subspace of $\mathcal{B} = \mathcal{D}_{L^\infty}$, defined as follows: $\varphi \in \mathcal{B}$ if and only if $|\varphi^{(\alpha)}(x)| \to 0$ as $\|x\| \to \infty$ for every $\alpha \in \mathbb{N}_0^n$.

$\mathcal{D}'_{L^p}, 1 < p \leq \infty$ is the dual space of $\mathcal{D}_{L^q}, 1 \leq q < \infty, \frac{1}{p} + \frac{1}{q} = 1$. $\mathcal{D}'_{L^p}$ is the dual of $\mathcal{B}$ and $\mathcal{D}'_{L^\infty}$ is denoted by $\mathcal{B}'$.

$\mathcal{O}'_c$ is the space of distributions with fast descent:

$\mathcal{O}'_c = \{ T \in \mathcal{D}'; (1 + |x|^2)^m T \in \mathcal{B}', \text{for every } m \in \mathbb{N} \}$.

$\mathcal{K}_{\nu}, p \geq 1$, is the spaces of functions $\varphi \in C^\infty$ with the property:

$$\nu_m(\varphi) = \sup_{x \in \mathbb{R}^n, |\alpha| \leq m} \exp(m\|x\|^p)|D^\alpha \varphi(x)| < \infty, \quad m = 1, 2, \ldots.$$
The elements of $K_p$ are called rapidly exponentially decreasing functions. Then $K'_p$ is the dual of $K_p$.

The convolution in $D'$ is defined as follows. Let $\langle \eta_m \rangle_{m \in \mathbb{N}}$ be a sequence in $D'(\mathbb{R}^n)$ such that for every compact set $K \subset \mathbb{R}^n$ there exists $m_0(K)$ such that $\eta_m(x) = 1$, $x \in K$, $m \geq m_0(K)$ and

$$\sup_{x \in \mathbb{R}^n} |\eta_m^{(\beta)}(x)| < C_\beta, \quad \beta \in \mathbb{N}_0^n.$$

The convolution of $T, S \in D'$ is defined by

$$\langle T * S, \varphi \rangle = \lim_{m \to \infty} \langle T(x)S(y), \eta_m(x,y)\varphi(x+y) \rangle, \quad \varphi \in D,$$

if this limit exists for every $\langle \eta_m \rangle_m$ (then, it does not depend on $\langle \eta_m \rangle_m$). By the Banach–Steinhaus theorem we know that $T * S \in D'$.

The spaces $D'(\Gamma)$ and $S'(\Gamma)$ with the operation $*$ are associative and commutative algebras. The convolution in this case is separately continuous.

We refer also to [63] and [3] for the theory of distributions.

0.5.2 Ultradistribution spaces

We follow the notation and definitions from [79], [81] and [86]. By $\langle M_p \rangle_p$ is denoted a sequence of positive numbers, $M_0 = M_1 = 1$, satisfying some of the following conditions:

(M.1) $M_p^2 \leq M_{p-1}M_{p+1}$, $p \in \mathbb{N};$
(M.2) $M_p/(M_qM_{p-q}) \leq AB^p$, $0 \leq q \leq p$, $p \in \mathbb{N};$
(M.2)' $M_{p+1} \leq AB^pM_p$, $p \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\};$
(M.3) $\sum_{q=p+1}^{\infty} M_{q-1}/M_q \leq A_pM_p/M_{p+1}$, $p \in \mathbb{N};$
(M.3)' $\sum_{p=1}^{\infty} M_{p-1}/M_p < \infty$,

where $A$ and $B$ are constants independent of $p$.

In the sequel, we will always assume (M.1), (M.2)' and (M.3)'.

Let $h$ be a positive number and let $\langle h_p \rangle_p$ be a real positive sequence increasing to $\infty$. We denote

$$H_p = \begin{cases} h^p, & \text{for the ultradifferentiable functions of class } (M_p) \\ h_1 \ldots h_p, & \text{for the ultradifferentiable functions of class } \{M_p\}. \end{cases}$$
Let $K$ be a compact set in an open set $\Omega$ of $\mathbb{R}^n$.

We denote by $E^{H_{p,M}}_K$ the space of smooth functions $\varphi \in E$ such that

$$\sup_{x \in K, \alpha \in \mathbb{N}^n_0} \frac{||\varphi^{(\alpha)}(x)||}{H_{|\alpha|}M_{|\alpha|}} < \infty.$$ (0.5)

We set $D^{H_{p,M}}_K = \{ \varphi \in E^{H_{p,M}}_K; \text{supp } \varphi \subset K \}$, it is a Banach space with norm

$$q_{H_{p,M}}(f) = \sup_{x \in K, \alpha \in \mathbb{N}^n_0} \frac{||\varphi^{(\alpha)}(x)||}{H_{|\alpha|}M_{|\alpha|}}.$$ (0.6)

Then, the basic spaces are defined by

$$D_{K}^{(M_p)} = \text{proj lim}_{h \to 0} D^{H_{p,M}}_K,$$
$$D_{K}^{\{M_p\}} = \text{ind lim}_{\{h \to 0\}} D^{H_{p,M}}_K.$$

where $*$ denotes either $(M_p)$ or $\{M_p\}$.

The spaces with the upper index $(M_p)$ are the Beurling-type spaces of ultradifferentiable functions and with the upper index $\{M_p\}$ are the Roumieu-type spaces of ultradifferentiable functions. Their strong duals are spaces of Beurling and Roumieu-type ultradistributions, respectively.

The space $E^{**} = E^{**}(\Omega)$ is the dual of

$$E^{(M_p)} = \text{proj lim}_{K \subset \subset \Omega} \text{proj lim}_{h \to 0} E^{H_{p,M}}_K,$$
$$E^{(M_p)} = \text{proj lim}_{K \subset \subset \Omega} \text{ind lim} E^{H_{p,M}}_K.$$

Weighted ultradistribution spaces are defined by

$$D_{L^1}^{(M_p)}(\mathbb{R}^n) = D_{L^1}^{(M_p)} = \text{proj lim}_{h \to \infty} D_{L^1,h}^{M_p},$$
$$D_{L^1}^{(M_p)} = \text{ind lim} D_{L^1,h}^{M_p},$$

where $D_{L^1,h}, h > 0$, is the Banach space of smooth functions $\varphi$ on $\mathbb{R}^n$ with finite norm

$$||\varphi||_{L^1,h} = \sup_{\alpha \in \mathbb{N}^n_0} \frac{h_{|\alpha|}}{M_{|\alpha|}||\varphi^{(\alpha)}||_{L^1}}.$$ $D^* = D^*(\mathbb{R}^n)$ is dense in $D_{L^1,1}^*$, and the inclusion mapping is continuous. The strong dual of $D_{L^1,1}^*$ is denoted by $B^{**}$.

Spaces of tempered ultradistributions are defined as the strong duals of the following testing function spaces:

$$S^{(M_p)}(\mathbb{R}^n) = S^{(M_p)} = \text{proj lim}_{h \to \infty} S_{h}^{M_p},$$
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$\mathcal{S}(\mathbb{M}_n)(\mathbb{R}^n) \equiv \mathcal{S}(\mathbb{M}_n) = \operatorname{ind\, lim}_{h \to 0} \mathcal{S}^M_h$, where $\mathcal{S}^M_h$, $h > 0$, is the Banach space of smooth functions $\varphi$ on $\mathbb{R}$ with finite norm

$$
\gamma_h(\varphi) = \sup_{\alpha, \beta \in \mathbb{N}_0^n} \frac{h^{\vert \alpha \vert + \vert \beta \vert}}{M_{\vert \alpha \vert}M_{\vert \beta \vert}} \| (1 + |x|^2)^{\vert \alpha \vert / 2} \varphi^{(\beta)} \|_{L^\infty}.
$$

The Fourier transform is an isomorphism of $\mathcal{S}^*$ onto $\mathcal{S}^*$; $\mathcal{D}^* = \mathcal{D}^*(\mathbb{R}^n)$ is dense in $\mathcal{S}^*$, $\mathcal{S}^*$ is dense in $\mathcal{D}^*$ and the inclusion mappings are continuous. The strong dual of $\mathcal{S}^*$, $\mathcal{S}^{\ast\ast}$, is the space of tempered ultradistributions (of Beurling and Roumieu types). There holds $\mathcal{D}^* \hookrightarrow \mathcal{S}^* \hookrightarrow \mathcal{E}^*$, where $\hookrightarrow$ means that the left space is dense in the right one and that the inclusion mapping is continuous. Thus, $\mathcal{E}^* \subset \mathcal{S}^* \subset \mathcal{D}^*$. We denote $\mathcal{S}^{\ast\ast}_+ = \{ f \in \mathcal{S}^{\ast\ast}(\mathbb{R}); \operatorname{supp} f \subset [0, \infty) \}$.

Let

$$
\mathcal{S}^{\ast\ast}_{(0, \infty)} = \{ \psi \in C^\infty(0, \infty); \psi = \varphi|_{[0, \infty)} \text{ for some } \varphi \in \mathcal{S}^* \}
$$

with the induced convergence structure from $\mathcal{S}^*$; its strong dual is in fact $\mathcal{S}^{\ast\ast}_+$. An operator of the form: $P(D) = \sum_{\vert \alpha \vert = 0}^\infty a_{\alpha} D^\alpha$, $a_{\alpha} \in C$, $\alpha \in \mathbb{N}_0^n$, is called ultradifferential operator of class $(\mathbb{M}_p)$ (of class $\{\mathbb{M}_p\}$) if there are constants $L > 0$ and $C > 0$ (for every $L > 0$ there is $C > 0$) such that $|a_{\alpha}| \leq C L^{\vert \alpha \vert} M_{\vert \alpha \vert}$, $\alpha \in \mathbb{N}_0$.

0.5.3. Fourier hyperfunctions. There are many equivalent definitions of hyperfunctions, Laplace hyperfunctions and Fourier hyperfunctions (cf. [71], [80], [144], [145], [204]), but we will use definitions and results collected in [75]. Let $I$ be a convex neighborhood of zero in $\mathbb{R}^n$ and let $\alpha$ be a non-negative constant. A function $F$, holomorphic on $\mathbb{R}^n + i I$, is said to decrease exponentially with type $(-\alpha)$, $\alpha \geq 0$, if for every compact subset $K \subset \subset I$ and every $\varepsilon > 0$, there exists $C_{K, \varepsilon} > 0$ such that

$$
|F(z)| \leq C_{K, \varepsilon} \exp\left( -\alpha \varepsilon |\operatorname{Re} z| \right), \quad z \in \mathbb{R}^n + i K.
$$

The set of all such functions is denoted by $\tilde{\mathcal{O}}^{-\alpha}(\mathbb{D}^n + i I)$, $(\tilde{\mathcal{O}}(\mathbb{D}^n + i I)$ for $\alpha = 0$), where $\mathbb{D}^n$ denotes the directional compactification of $\mathbb{R}^n$: $\mathbb{D}^n = \mathbb{R}^n \cup \mathbb{S}^{n-1}(\mathbb{S}^n\infty$ consists of all points at infinity in all direction). Space $\mathcal{P}_\ast$ is defined by

$$
\mathcal{P}_\ast = \operatorname{ind\, lim}_{I \ni 0} \operatorname{ind\, lim}_{\alpha \ni 0} \tilde{\mathcal{O}}^{-\alpha}(\mathbb{D}^n + i I).
$$
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The dual space of \( \mathcal{P}_* \) is the space \( \mathcal{Q}(\mathbb{D}^n) \) of the Fourier hyperfunctions. \( \mathcal{Q}(\mathbb{D}^n) \) is a space of FS-type. We can give a representation of elements from \( \mathcal{Q}(\mathbb{D}^n) \). Let \( \mathcal{O} \) be the sheaf of holomorphic functions. Denote \( U_j = (\mathbb{D}^n + iI) \cap \{ \text{Im} z_j \neq 0 \} \), \( j = 1, \ldots, n \); \( (\mathbb{D}^n + iI)D^n = U_1 \cap \cdots \cap U_n \),

\[
(\mathbb{D}^n + iI)\#D^n = U_1 \cap \cdots \cap U_{j-1} \cap U_{j+1} \cap \cdots \cap U_n.
\]

Then

\[
\mathcal{Q}(\mathbb{D}^n) = \hat{\mathcal{O}}((\mathbb{D}^n + iI)\#D^n)/\sum_{j=1}^n \hat{\mathcal{O}}((\mathbb{D}^n + iI)\#D^n). \quad (0.9)
\]

Thus, \( f \in \mathcal{Q}(\mathbb{D}^n) \) is defined as the class \([F] \), where \( F \in \hat{\mathcal{O}}((\mathbb{D}^n + iI)\#D^n) \). \( F \) is called a defining function of \( f \) and it is represented by \( 2^n \) functions \( F_\sigma, F = [F_\sigma] \), where \( F_\sigma \in \hat{\mathcal{O}}(\mathbb{D}^n + iI_\sigma) \); \( \mathbb{D}^n + iI_\sigma = \mathbb{D}^n + i(I \cap \Gamma_\sigma) \) is an infinitesimal wedge of type \( \mathbb{R}^n + i\Gamma_\sigma0 \), \( \Gamma_\sigma \) are open \( \sigma \)-th orthants in \( \mathbb{R}^n \).

An \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) is called a function of infra-exponential type if for every \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that \(|f(x)| \leq C_\varepsilon \exp(\varepsilon|x|) \), \( x \in \mathbb{R}^n \). Then by \( \ell f \) is denoted the hyperfunction defined by \( f \).

We denote by \( \Lambda \) the set of \( n \)-th variations of \( \{-1, 1\} \).

The boundary-value representation of \( f \in \mathcal{Q}(\mathbb{D}^n) \) is:

\[
f = [F] := \sum_{\sigma \in \Lambda} F_\sigma(x + i\Gamma_\sigma0). \quad (0.10)
\]

\( F_\sigma(x + i\Gamma_\sigma0) \) denotes the element of the quotient space given in (0.9); it is determined by \( F_\sigma \).

The dual pairing between \( \varphi \in \mathcal{P}_* \) and \( g = [G] \in \mathcal{Q}(\mathbb{D}^n) \) is given by

\[
\langle g, \varphi \rangle = \int_{\mathbb{R}^n} g(x)\varphi(x)dx = \sum_{\sigma \in \Lambda} \int_{\text{Im } w = v_\sigma} G_\sigma(w)\varphi(w)dw, \quad w = u + iv,
\]

where \( v_\sigma \in I_\sigma \).

Similarly, \( \mathcal{Q}^{-\alpha}(\mathbb{D}^n) \), \( \alpha > 0 \), is defined using \( \hat{\mathcal{O}}^{-\alpha} \) instead of \( \hat{\mathcal{O}} \) (cf. Definition 8.2.5 in [75]).

An infinite-order differential operator \( J(D) \) of the form

\[
J(D) = \sum_{|\alpha| \geq 0} b_\alpha D^\alpha \quad \text{with} \quad \lim_{|\alpha| \to \infty} |\sqrt{\alpha!} b_\alpha| = 0, b_\alpha \in \mathbb{C},
\]

is called a local operator. \( J(D) \) maps continuously \( \mathcal{O}(U) \) into \( \mathcal{O}(U) \), \( U \) being an open set in \( \mathbb{C}^n \), and also \( \mathcal{Q}(\mathbb{D}^n) \) into \( \mathcal{Q}(\mathbb{D}^n) \).
The Fourier transform on $\mathcal{Q}(\mathbb{D}^n)$ is defined by using the functions $\chi_\sigma = \chi_{\sigma_1}(z_1)\ldots\chi_{\sigma_n}(z_n)$, where $\sigma_k = \pm 1$, $k = 1, \ldots, n$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\chi_+ (t) = e^t/(1 + e^t)$, $\chi_- (t) = 1/(1 + e^t)$, $t \in \mathbb{R}$. Let

$$u(x) \equiv \sum_{\sigma \in \Lambda} U_\sigma (x + i\Gamma_\sigma 0) = \sum_{\sigma \in \Lambda} \sum_{\sigma \in \Lambda} (\chi_\sigma U_\sigma)(x + i\Gamma_\sigma 0),$$

where $\chi_\sigma U_\sigma \in \mathcal{Q}(\mathbb{D}^n + iI_\sigma)$, $\hat{\sigma} \in \Lambda$ and $\chi_\sigma U_\sigma$ decreasing exponentially along the real axis outside the closed $\hat{\sigma}$-th orthant. The Fourier transform of $u$ is defined by

$$\mathcal{F}(u)(\xi) \equiv \sum_{\sigma \in \Lambda} \sum_{\sigma \in \Lambda} \mathcal{F}(\chi_\sigma U_\sigma)(\xi - i\Gamma_\sigma 0) = \sum_{\sigma \in \Lambda} \sum_{\sigma \in \Lambda} \int e^{iz\xi} (\chi_\sigma U_\sigma)(z) dz, \quad y^\sigma \in I_\sigma, \quad \xi = \im \eta,$$

where $\mathcal{F}(\chi_\sigma U_\sigma) \in \mathcal{O}(\mathbb{D}^n - iI_\sigma)$ and $\mathcal{F}(\chi_\sigma U_\sigma)(z) = O(e^{-\omega|z|})$ for a suitable $\omega > 0$ along the real axis outside the closed $\sigma$-orthant. $\mathcal{F}$ is an automorphism of $\mathcal{Q}(\mathbb{D}^n)$.

0.6. Let $E$ be a locally convex topological vector space. An absolute convex closed absorbent subset of $E$ is called a barrel. If every barrel is a neighborhood of zero in $E$, then $E$ is called barrelled. Throughout this book $\mathcal{F}_g$ stands for a locally convex barrelled complete Hausdorff space of smooth functions (the subscript $g$ stands for “general”), $\mathcal{F}_g \hookrightarrow \mathcal{E} = \mathcal{E}(\mathbb{R}^n)$, and $\mathcal{F}_g'$ stands for the strong dual space of $\mathcal{F}_g$; observe $\mathcal{E}' \subset \mathcal{F}_g'$. If $T \in \mathcal{F}_g'$ and $\varphi \in \mathcal{F}_g$, then $(T, \varphi)$ is the dual pairing between $T$ and $\varphi$. We write $\mathcal{F}_0$ if all elements of $\mathcal{F}_g$ are compactly supported; $\mathcal{F}_g'$ denotes the dual space of $\mathcal{F}_0$; observe that a notion of support for elements of $\mathcal{F}_0$ can be defined in the usual way. In $\mathcal{F}_g'$, a weakly bounded set is also a strongly bounded one (Mackey–Banach–Steinhaus theorem). The spaces of distributions, ultradistributions, Fourier hyperfunctions, . . . , are of this kind. Furthermore, we shall always use the notation $\mathcal{F} = \mathcal{F}_g$ if $A \hookrightarrow \mathcal{F}$, where $A = \mathcal{D}, \mathcal{D}^\ast$, or $\mathcal{P}_*$; in such case $\mathcal{F}' \subset \mathcal{A}' = \mathcal{D}', \mathcal{D}^\ast'$, or $\mathcal{Q}(\mathbb{D}^n)$, respectively, and we say that $\mathcal{F}'$ is a distribution, ultradistribution, or Fourier hyperfunction space, respectively. We set $\mathcal{F}_T^\prime = \{ T \in \mathcal{F}'; \supp T \subset \Gamma \}$.

We suppose that the following operations are well defined on $\mathcal{F}_g'$:

**Differentiation:** We assume that $\partial/\partial x_i$, $\mathcal{F}_g \to \mathcal{F}_g$, are continuous operators. Let $k \in \mathbb{N}_0^n$. Then,

$$\left\langle \frac{\partial^k}{\partial x_1 \ldots \partial x_n} T(x), \varphi(x) \right\rangle = \left\langle T(x), (-1)^{|k|} \frac{\partial^k}{\partial x_1 \ldots \partial x_n} \varphi(x) \right\rangle, \quad \varphi \in \mathcal{F}_g.$$
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Change of variables: If $T \in \mathcal{F}_g'$ and $k > 0$, then by definition
$$\langle T(kx), \varphi(x) \rangle = \left\langle T(x), \frac{1}{k^n} \varphi\left(\frac{x}{k}\right) \right\rangle, \quad \varphi \in \mathcal{F}_g.$$

If $T \in \mathcal{F}_g'$ and $h \in \mathbb{R}^n$,
$$\langle T(x + h), \varphi(x) \rangle = \left\langle T(x), \varphi(x - h) \right\rangle, \quad \varphi \in \mathcal{F}_g.$$

Furthermore, it is always assumed that $k \to \varphi(\cdot/k)$, $\mathbb{R}_+ \to \mathcal{F}_g$, and $h \to \varphi(\cdot + h)$, $\mathbb{R}^n \to \mathcal{F}_g$, are continuous. Consequently, by the mean value theorem, one readily verifies that both maps are indeed $C^\infty$.

Let $\psi \in \mathcal{E}$, if $\varphi \mapsto \psi\varphi$ is a continuous mapping from $\mathcal{F}_g'$ into $\mathcal{F}_g$, then we say that $\psi$ is a multiplier of $\mathcal{F}_g'$. Then $\psi T$ is by definition $\langle \psi T, \varphi \rangle = \langle T, \psi \varphi \rangle$.

The set of multipliers of $\mathcal{F}_g'$ is denoted by $M(\cdot)$.

We shall say that $\theta \in \mathcal{E}$ is a convolutor of $\mathcal{F}_g'$ if the mapping $\varphi \mapsto \hat{\theta} \ast \varphi$, $\mathcal{F}_g \to \mathcal{F}_g$, is well defined and continuous, where $\hat{\theta}(t) = \theta(-t)$. We denote by $M_\ast$ the set of convolutors of $\mathcal{F}_g'$, if $\theta \in M_\ast$, then for $T \in \mathcal{F}_g'$, $(T \ast \theta)$ is defined by $\langle T \ast \theta, \psi \rangle = \langle T, \hat{\theta} \ast \psi \rangle$, $\varphi \in \mathcal{F}_g$.

$(\delta_m)_{m \in \mathbb{N}}$, a sequence in $M_\ast \cap \mathcal{F}_g$, is called a $\delta$-sequence in $\mathcal{F}_g'$ if $\delta_m \geq 0$, $m \in \mathbb{N}$, and for every $\varphi \in \mathcal{F}_g$, $\delta_m \ast \varphi \to \varphi$ in $\mathcal{F}_g$, $m \to \infty$.

We shall say that the convolution with compactly supported elements is well defined in $\mathcal{F}_g'$ if a notion of support makes sense in $\mathcal{F}_g'$ and the following definition applies: given $S, T \in \mathcal{F}_g'$, where supp $S = K$ is compact in $\mathbb{R}^n$, their convolution is defined by
$$\langle S \ast T, \varphi \rangle = \langle S(x) \times T(y), \alpha(x) \varphi(x + y) \rangle = \langle S_x \times T_y, \alpha(x) \varphi(x + y) \rangle,$$

where the function $\alpha \in \mathcal{F}_g$ has compact support, supp $\alpha = \bar{K}$, so that the set $K \subset \text{int} \bar{K}$ and $\alpha(x) = 1$, $x \in K$. For $\mathcal{F}'$ it coincides with the usual definition of convolution.

We assume that $\mathcal{F}_g'$ contains regular elements $f \in L^1_{\text{loc}}$. They are identified with $f$ itself:
$$f : \langle f, \varphi \rangle = \int f(x) \varphi(x) dx, \quad \varphi \in \mathcal{F}_g,$$

if $f \varphi \in L^1$ for every $\varphi \in \mathcal{F}_g$ and if $\varphi_m \to 0$ in $\mathcal{F}$ implies $\langle f, \varphi_m \rangle \to 0$.

There is also another situation in which we shall identify locally integrable functions with elements $T \in \mathcal{F}'$. Let $\mathcal{A} = \mathcal{D}, \mathcal{D}'$, or $\mathcal{P}$, and suppose $\mathcal{A} \hookrightarrow \mathcal{F}$. We identify $T$ with $f \in L^1_{\text{loc}}$ if $\langle T, \varphi \rangle = \langle f, \varphi \rangle$, for all $\varphi \in \mathcal{A}$, in such a case we simply write $T = f$. For example, $T(x) = xe^{-x^2} \sin(e^x^2) \in \mathcal{S}'(\mathbb{R})$ is defined in this way, and not as a regular element of $\mathcal{S}'(\mathbb{R})$ in the sense described above.
1 S-asymptotics in $\mathcal{F}_g'$

1.1 Definition

Definition 1.1. Let $\Gamma$ be a cone with vertex at zero and let $c$ be a positive real-valued function defined on $\Gamma$. It is said that $T \in \mathcal{F}_g'$ has S-asymptotic behavior related to $c$ with limit $U$ if $T(x+h)/c(h)$ converges weakly in $\mathcal{F}_g'$ to $U$ when $h \in \Gamma$, $\|h\| \to \infty$, i.e. (w. lim = weak limit):

$$\text{w. lim}_{h \in \Gamma, \|h\| \to \infty} T(x+h)/c(h) = U \quad \text{in} \quad \mathcal{F}_g' \quad (1.1)$$

or

$$\lim_{h \in \Gamma, \|h\| \to \infty} \langle T(x+h)/c(h), \varphi(x) \rangle = \langle U, \varphi \rangle, \quad \varphi \in \mathcal{F}_g. \quad (1.2)$$

If (1.1) is satisfied, it is also said that $T$ has S-asymptotics and we write in short:

$$T(x+h) \overset{s}{\sim} c(h)U(x), \quad h \in \Gamma.$$  

Remarks. 1) If $\Gamma$ is a convex cone we could use another limit in $\Gamma$. Let $h_1, h_2 \in \Gamma$. We say that $h_1 \geq h_2$ if and only if $h_1 \in h_2 + \Gamma$; $\Gamma$ is now partially ordered.

For a real-valued function $\rho$ defined on $\Gamma$, we write

$$\lim_{h \in \Gamma, h \to \infty} \rho(h) = A \in \mathbb{R}$$

if for any $\varepsilon > 0$ there exists $h(\varepsilon) \in \Gamma$ such that $\rho(h) \in (A - \varepsilon, A + \varepsilon)$ when $h \geq h(\varepsilon)$, $h \in \Gamma$.

If $\Gamma$ is a convex cone, then the S-asymptotics with respect to this limit might be defined as:

$$\lim_{h \in \Gamma, h \to \infty} \langle T(x+h)/c(h), \varphi(t) \rangle = (u, \varphi), \quad \varphi \in \mathcal{F}_g. \quad (1.3)$$

In case $n = 1$, the limits (1.2) and (1.3) coincide. We will mostly use Definition 1.1 in this book, for S-asymptotics defined by (1.3) see also [135].

2) If $\mathcal{F}_g$ is a Montel space, then the strong and the weak topologies in $\mathcal{F}_g'$ are equivalent on a bounded set. If $B$ is a filter with a countable basis and if $w. \lim_{h \in B} T(h) = U$ in $\mathcal{F}_g'$, then this limit exists in the sense of the strong topology. Hence, (1.1) is equivalent to

$$\text{s. lim}_{h \in \Gamma, \|h\| \to \infty} T(x+h)/c(h) = U \quad \text{in} \quad \mathcal{F}_g'.$$
(Furthermore, from now on, we will omit the symbol $s$. for the strong convergence).

For the first ideas of the $S$-asymptotics see [3] and [146]. The starting point of the theory is [127].

### 1.2 Characterization of comparison functions and limits

**Proposition 1.1.** Let $\Gamma$ be a convex cone. Suppose $T \in \mathcal{F}'_g$ has the $S$-asymptotics $T(x + h) \sim c(h)U(x)$, $h \in \Gamma$. If $U \neq 0$, then:

**a)** There exists a function $d$ on $\Gamma$ such that

$$
\lim_{h \in \Gamma, \|h\| \to \infty} \frac{c(h + h_0)}{c(h)} \frac{c(h)}{c(h + h_0)} = d(h_0), \text{ for every } h_0 \in \Gamma.
$$

**b)** The limit $U$ satisfies the equation

$$
U(\cdot + h) = d(h)U, \ h \in \Gamma.
$$

**Proof.**

**a)** Since $U \neq 0$, there exists a $\tilde{\varphi} \in \mathcal{F}_g$ such that $\langle U, \tilde{\varphi} \rangle \neq 0$. For this $\tilde{\varphi}$ and a fixed $h_0 \in \Gamma$

$$
\lim_{h \in \Gamma, \|h\| \to \infty} \frac{c(h + h_0)}{c(h)} \frac{c(h)}{c(h) + h_0} \langle T(x + (h + h_0)), \tilde{\varphi}(x) \rangle = d(h_0).
$$

Hence, for every $h_0 \in \Gamma$

$$
\lim_{h \in \Gamma, \|h\| \to \infty} \frac{c(h + h_0)}{c(h)} = \frac{\langle U(x + h_0), \tilde{\varphi}(x) \rangle}{\langle U, \tilde{\varphi} \rangle} = d(h_0).
$$

**b)** Now we can take in (1.5) any function $\varphi \in \mathcal{F}_g$ instead of $\tilde{\varphi}$. Then we have

$$
d(h_0)(U, \varphi) = \langle U(x + h_0), \varphi \rangle, \ \varphi \in \mathcal{F}_g
$$

which proves b). \hfill \Box

We now restrict the space of generalized functions to a space of distribution, ultradistribution, or Fourier hyperfunction type. So, we have the following explicit characterization of the comparison function and limit.
Proposition 1.2. Let $\Gamma$ be a convex cone with $\text{int} \, \Gamma \neq \emptyset$ (int $\Gamma$ is the interior of $\Gamma$). Let $T \in \mathcal{F}^i$ have $S$-asymptotics $T(x + h) \sim c(h)U(x)$, $h \in \Gamma$, where $U \neq 0$ and $c$ is a positive function defined on $\mathbb{R}^n$. Then:

a) For every $h_0 \in \mathbb{R}^n$ there exists
$$\lim_{h \in (h_0 + \Gamma) \cap \Gamma, \|h\| \to \infty} c(h + h_0)/c(h) = \tilde{d}(h_0).$$

b) There exists $\alpha \in \mathbb{R}^n$ such that $\tilde{d}(x) = \exp(\alpha \cdot x)$, $x \in \mathbb{R}^n$.

c) There exists $C \in \mathbb{R}$ such that $U(x) = C \exp(\alpha \cdot x)$.

Proof. a) Let $a \in \text{int} \, \Gamma$. Then there exists $r > 0$ such that $B(a, r) \subset \Gamma$. Consequently, for every $\beta > 0$, $B(\beta a, \beta r) \subset \Gamma$, as well.

We shall prove that for every $h_0 \in \mathbb{R}^n$ and every $R > 0$ the set $(h_0 + \Gamma) \cap \Gamma \cap \{x \in \mathbb{R}^n; \|x\| > R\}$ is not empty. The first step is to prove that $(h_0 + \Gamma) \cap \Gamma$ is not empty.

Suppose that $y \in B(a, r/2) \subset \Gamma$. Then, for every $\beta \geq \beta_0 > 2\|h_0\|/r > 0$, $\|\beta a - (h_0 + \beta y)\| \leq \beta\|a - y\| + \|h_0\| \leq \beta r$, hence $h_0 + \beta y \in B(\beta a, \beta r) \subset \Gamma$.

For a fixed $R > 0$, we can choose $\beta$ such that $\|h_0 + \beta y\| > R$. Then $h_0 + \beta y$ is a common element for $(h_0 + \Gamma), \Gamma$ and $\{x \in \mathbb{R}^n; \|x\| > R\}$.

Now we can use the limit (1.5) when $h \in (h_0 + \Gamma) \cap \Gamma$, and in the same way as in the proof of Proposition 1.1 a), we obtain
$$\lim_{h \in (h_0 + \Gamma) \cap \Gamma, \|h\| \to \infty} \frac{c(h + h_0)}{c(h)} = \tilde{d}(h_0) = \frac{(U(t + h_0), \tilde{\varphi})}{(U, \tilde{\varphi})}.$$

From the existence of this limit, it follows:

1) $\tilde{d}$ extends $d$ to the whole $\mathbb{R}^n$;

2) $\tilde{d}(0) = 1$; $\tilde{d} \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\tilde{d}$ satisfies
$$\tilde{d}(h + h_0) = \tilde{d}(h)d(h_0), \quad h, h_0 \in \mathbb{R}^n. \quad (1.6)$$

We can take $h_0 = (0, \ldots, 0, t_i, 0, \ldots) \in \mathbb{R}^n$ in (1.6); then the limit
$$\lim_{t_i \to 0} \frac{\tilde{d}(h + h_0) - \tilde{d}(h)}{t_i} = \tilde{d}(h) \lim_{t_i \to 0} \frac{\tilde{d}(h_0) - \tilde{d}(0)}{t_i}$$
exists and gives
$$\frac{\partial}{\partial h_i} \tilde{d}(h) = \left( \frac{\partial}{\partial h_i} \tilde{d}(h) \right)_{h=0} \tilde{d}(h), \quad \text{for } i = 1, \ldots, n.$$
We introduce the function $V$ given by
\[ \tilde{d}(h) = e^{(\alpha \cdot h)} V(h), \quad \alpha_i = \left( \frac{\partial}{\partial h_i} \tilde{d}(h) \right)_{h=0}, \quad i = 1, \ldots, n. \]
Then $\frac{\partial}{\partial h_i} V(h) = 0$ for every $i = 1, \ldots, n$. Consequently, $V(h) = 1$, $h \in \mathbb{R}^n$.

For the proof of c), we now have $U(x+h) = \exp(\alpha \cdot h) U(x)$. Differentiating with respect to $h$ and then setting $h = 0$, we obtain that $U$ satisfies the differential equations $\frac{\partial}{\partial x_j} U = \alpha_j$, $1 \leq j \leq n$. Proceeding as in the proof of b), we obtain $U(x) = C \exp(\alpha \cdot x)$, for some $C \in \mathbb{R}$. □

Remarks.

1. We only assumed that $c$ is a positive function. But if we know that there exist $T \in \mathcal{F}'_g$ and $U \neq 0$ such that $T(x+h) \sim c(h) U(t)$, $h \in \Gamma$, then we can find a function $\tilde{c} \in C^\infty$ and with the property
\[ \lim_{h \in \Gamma, \|h\| \to \infty} \tilde{c}(h)/c(h) = 1. \]
This function $\tilde{c}$ can be defined as follows: $\tilde{c}(h) = \langle T(x+h), \tilde{\varphi}(x) \rangle / \langle U, \tilde{\varphi} \rangle$, where $\tilde{\varphi}$ is chosen so that $\langle U, \tilde{\varphi} \rangle \neq 0$. In this sense we can suppose, whenever needed, that $c \in C^\infty$, and we do not lose generality.

Similarly, we have (see also Lemma 1.4 in 1.12) that
\[ \lim_{h \in \Gamma, \|h\| \to \infty} c(h)/\tilde{c}(h+x) = \exp(-\alpha \cdot x) \quad \text{in } \mathcal{E} \]
if $\Gamma$ is a convex cone, $\text{int } \Gamma \neq \emptyset, \Gamma' \subset \subset \Gamma$ (\$\Gamma'\$ is compact int $\Gamma$).

2. If in Proposition 1.2 we replace $\mathcal{F}'$ for the general space $\mathcal{F}'_g$, then a) and b) still hold. On the other hand, c) will not be longer true, in general. We will show this fact in Remark 5 below by constructing explicit counterexamples.

3. In the one-dimensional case the cone $\Gamma$ can be only $\mathbb{R}, \mathbb{R}_+$ or $\mathbb{R}_-$. In all these three cases int $\Gamma \neq \emptyset$. Consequently, $\tilde{d}$ from Proposition 1.2 has always the form $\tilde{d}(x) = \exp(\alpha x)$, where $\alpha \in \mathbb{R}$.

Let us write $c(x) = L(e^x) \exp(\alpha x)$, $x \in \mathbb{R}$. We will show that $L$ is a slowly varying function. Proposition 1.2 a) gives us the existence of the limit
\[ \lim_{h \in \Gamma, \|h\| \to \infty} L(\exp(h+h_0))/L(\exp(h)) = 1, \quad h_0 \in \mathbb{R}. \]
If $\Gamma = \mathbb{R}_+$, then
\[ \lim_{x \to \infty} \frac{L(xp)}{L(x)} = 1, \quad p \in \mathbb{R}_+ \]
and this defines a slowly varying function (cf. 0.3). Thus if $T \in \mathcal{F}_g'(\mathbb{R})$ and $T(x+h) \sim c(h)U(x)$, in $\mathbb{R}_+$, with $U \neq 0$, then it follows that $c$ has the form $c(x) = \exp(\alpha x)L(\exp(x))$, $x \geq a > 0$, where $L$ is a slowly varying function at infinity. Similarly, if $\Gamma = \mathbb{R}_-$, then $L$ is slowly varying at the origin, while if $\Gamma = \mathbb{R}$, then $L$ is slowly varying at both infinity and the origin.

4. The explicit form of the function $c$ given in 3. is not known in the $n$-dimensional case, $n \geq 2$. This problem is related to the extension of the definition of a regularly varying function to the multi-dimensional case ([135], [162]) and with certain $q$-admissible and $q$-strictly admissible functions ([192]).

5. As mentioned before, c) in Proposition 1.2 does not have to hold for a general space $\mathcal{F}_g'$. From the proof of Proposition 1.2, we can still obtain the weaker conclusion $U(x+h) = \exp(\alpha x)U(x)$, $h \in \mathbb{R}^n$, which in turn implies the differential equations $(\partial/\partial x_j)U = \alpha_j$. For distribution, ultradistribution, and Fourier hyperfunction spaces, these differential equations imply that $U$ must have the form c) of Proposition 1.2. However, the latter fact is not true in general. We provide two related counterexamples below.

Let
\[ \mathcal{A}_0(\mathbb{R}) = \{ \varphi \in C^\infty(\mathbb{R}); \lim_{x \to -\infty} \varphi^{(m)}(x) \text{ exists and is finite}, m \in \mathbb{N}_0 \}, \]
it is a Frechet space with seminorms:
\[ \beta_k(\varphi) = \sup_{x \in (-\infty,k]} |\varphi^{(m)}(x)|. \]
Let us first observe that the definition of $\mathcal{A}_0(\mathbb{R})$ does not tell all the true about its elements. Notice that if $\varphi \in \mathcal{A}_0(\mathbb{R})$, then for $m = 1, 2, \ldots$, we have $\lim_{x \to -\infty} \varphi^{(m)}(x) = 0$, while $\lim_{x \to -\infty} \varphi(x)$ may not be zero. Indeed, the proof is easy, it is enough for $m = 1$, if $\lim_{x \to -\infty} \varphi'(x) = M$, then
\[ \varphi(x) = \varphi(0) + \int_0^x \varphi'(t)dt \sim \varphi(0) + Mx, \quad x \to -\infty, \]
but since $\varphi$ has limit at $-\infty$, then $M = 0$. So, we have
\[ \mathcal{A}_0(\mathbb{R}) = \{ \varphi \in C^\infty(\mathbb{R}); \lim_{x \to -\infty} \varphi(x) \text{ exists and } \lim_{x \to -\infty} \varphi^{(m)}(x) = 0, m \in \mathbb{N} \}. \]
1. S-asymptotics in $\mathcal{F}_g'$

Its dual space $\mathcal{A}_0'(\mathbb{R})$ contains a generalized function concentrated at $-\infty$ which contradicts c) of Proposition 1.2. Define the Dirac delta concentrated at $-\infty$ by

$$(\delta_{-\infty}, \varphi) := \lim_{x \to -\infty} \varphi(x), \quad \varphi \in \mathcal{A}_0(\mathbb{R}).$$

Notice that the constant multiples of $\delta_{-\infty}$ are the only elements of $\mathcal{A}_0'(\mathbb{R})$ satisfying the differential equation $U' = 0$. This generalized function is translation invariant, i.e., $\delta_{-\infty}(\cdot + h) = \delta_{-\infty}$, for all $h \in \mathbb{R}$; in particular, it has the S-asymptotics

$$\delta_{-\infty}(x + h) \sim \delta_{-\infty}(x), \quad h \in \Gamma = \mathbb{R}.$$

Therefore, we have found an example of a non-constant limit for S-asymptotics related to the constant function $c(h) = 1$.

We can go beyond the previous example and give a counterexample for the failure of conclusion c) of Proposition 1.2 with a general $c$ and S-asymptotics in $\mathcal{F}_g'$. Let $c(h) = \exp(\alpha h)L(\exp h)$, where $L$ is slowly varying at infinity and $\alpha \in \mathbb{R}$. We assume that $c$ is $C^\infty$ and, for all $m \in \mathbb{N}$,

$$c^{(m)}(h) \sim \alpha^m c(h), \quad h \to \infty$$

(otherwise replace $c$ by $\tilde{c}$ given in Remark 1).

Next, we define

$$\mathcal{A}_c(\mathbb{R}) = \{ \varphi \in C^\infty(\mathbb{R}); \lim_{x \to -\infty} \frac{\varphi^{(m)}(x)}{c^{(-x)}} \text{ exists and is finite}, \ m \in \mathbb{N}_0 \}.$$  

It is a Frechet space with seminorms:

$$\beta_{k,c}(\varphi) = \sup_{x \in (-\infty,k], m \leq k} \frac{|\varphi^{(m)}(x)|}{c(-x)}.$$  

Note that $\mathcal{A}_c(\mathbb{R}) = c(-x) \cdot \mathcal{A}_0(\mathbb{R})$, consequently,

$$\varphi'(x) \sim -\alpha C_\varphi c(-x), \quad x \to -\infty,$$

where $C_\varphi = \lim_{x \to -\infty} \varphi(x)/c(-x)$. An inductive argument shows that for all $m$, $\varphi^{(m)}(x) \sim (-1)^m \alpha^m C_\varphi c(-x), \ x \to -\infty$. Set now $g_{c,-\infty} = (1/c(-x)) \cdot \delta_{-\infty} \in \mathcal{A}_c(\mathbb{R})$, a generalized function concentrated at infinity and given by

$$(g_{c,-\infty}, \varphi) = \left\langle \delta_{-\infty}(x), \frac{\varphi(x)}{c(-x)} \right\rangle = \lim_{x \to -\infty} \frac{\varphi(x)}{c(-x)}, \quad \varphi \in \mathcal{A}_c(\mathbb{R}).$$

It is easy to show that the constant multiples of $g_{c,-\infty}$ are the only elements of $\mathcal{A}_c'(\mathbb{R})$ satisfying the functional equation $U(x + h) = \exp(\alpha h)U(x)$ (and hence the differential equation $U' = \alpha U$); in particular,

$$g_{c,-\infty}(x + h) \sim e^{\alpha h} g_{c,-\infty}(x), \quad h \in \Gamma = \mathbb{R}.$$
In addition, there is an infinite number of elements of \(A'_c(\mathbb{R})\) having S-asymptotics in the cone \(\Gamma = \mathbb{R}_+\) related to \(c(h)\) with limit of the form \(Cg_{c,-\infty}(x)\), \(C \in \mathbb{R}\). In fact, consider \(\delta^{(m)}\), the derivatives of the Dirac delta concentrated at the origin. We have that
\[
\delta^{(m)}(x + h) \overset{s}{\sim} c(h)\alpha^m g_{c,-\infty}(x), \quad h \in \Gamma = \mathbb{R}_+,
\]
since
\[
\lim_{h \to \infty} \left\langle \delta^{(m)}(x + h), \varphi(x) \right\rangle = \lim_{h \to \infty} (-1)^m \frac{\varphi^{(m)}(-h)}{c(h)} = \alpha^m \left( g_{c,-\infty}(x), \varphi(x) \right).
\]

### 1.3 Equivalent definitions of the S-asymptotics in \(\mathcal{F}'\)

**Theorem 1.1.** Let \(T \in \mathcal{F}'\) and let \(\text{int} \Gamma \neq \emptyset\). The following assertions are equivalent:

a) \(w. \lim_{h \in \Gamma \|h\| \to \infty} \frac{T(x + h)}{c(h)} = U(x) = M \exp(\alpha x) \quad \text{in} \quad \mathcal{F}', \quad M \neq 0\). \hspace{1cm} (1.7)

b) For a \(\delta\)-sequence \(\langle \delta_m \rangle_m\) (cf. 0.6) there exists a sequence \(\langle M_m \rangle_m\) in \(\mathbb{R}\), such that \(M_m \to M \neq 0\), \(m \to \infty\), and
\[
w. \lim_{h \in \Gamma \|h\| \to \infty} \frac{(T * \delta_m)(x + h)}{c(h)} = M_m \exp(\alpha x), \quad \text{in} \quad \mathcal{F}', \quad \text{uniformly in} \quad m \in \mathbb{N}.
\]

(1.8)

c) For a \(\delta\)-sequence \(\langle \delta_m \rangle_m\), (cf. 0.6),
\[
\lim_{h \in \Gamma \|h\| \to \infty} \frac{(T * \delta_m)(h)}{c(h)} = p_m, \quad m \in \mathbb{N},
\]
where \(p_m \neq 0\) for some \(m\), and for every \(\phi \in \mathcal{F}\),
\[
\sup_{h \in \Gamma \|h\| \geq 0} \left| \frac{(T * \phi)(h)}{c(h)} \right| < \infty.
\]

(1.10)

If \(\mathcal{F}' = \mathcal{D}'\) or \(\mathcal{F}' = \mathcal{D}'^*\) the following assertion is also equivalent to a).

d) \((T(\cdot + h)/c(h)) * \varphi\) converges to \((U * \varphi)(h)\) in \(\mathcal{E}(\mathbb{R}^n)\), for each \(\varphi \in \mathcal{D} \quad (\varphi \in \mathcal{D}^*)\).
1. S-asymptotics in $\mathcal{F}^r_y$

Proof. \( a \Rightarrow b \). Let \( \{\delta_m\} \) be a \( \delta \)-sequence. For any \( \phi \in \mathcal{F} \), \( \{\delta_m \ast \phi; m \in \mathbb{N}\} \) is a compact set in \( \mathcal{F} \). We have by the properties of the convolution (cf. 0.6) and the Banach–Steinhaus theorem

$$
\lim_{h \to \infty} \frac{(T \ast \delta_m)(x + h)}{c(h)} = \frac{(T \ast \delta_m)(x)}{c(h)}
$$

uniformly in \( m \). Now (1.11) implies (1.8) and b).

\( b \Rightarrow a \). Let \( \phi \in \mathcal{F} \) and

$$
a_{m,h} = \left\langle \frac{(T \ast \delta_m)(x + h)}{c(h)}, \phi(x) \right\rangle = \frac{(T \ast \delta_m \ast \phi)(h)}{c(h)}, m \in \mathbb{N}, h \in \Gamma.
$$

We have \( a_{m,h} \to a_m, h \in \Gamma, \|h\| \to \infty \), uniformly for \( m \in \mathbb{N} \), where

$$
a_m = M_m(\exp(\alpha \cdot x), \phi(x), m \in \mathbb{N},
$$

$$
a_m \to a = M(\exp(\alpha \cdot x), \phi(x), m \to \infty.
$$

Also \( a_{m,h} \to a_h, m \to \infty \), where

$$
a_h = \left\langle \frac{T(x + h)}{c(h)}, \phi(x) \right\rangle, h \in \Gamma.
$$

This implies \( a_h \to a, h \in \Gamma, \|h\| \to \infty \), what is in fact a).

\( a \Rightarrow c \). From (1.7) it follows that \( (T \ast \phi)(h)/c(h) \) converges for every \( \varphi \in \mathcal{F} \), when \( h \in \Gamma, \|h\| \to \infty \). Hence, \( (T \ast \varphi)(h)/c(h) \) is bounded, \( h \in \Gamma, \|h\| \geq 0; \) (1.9) follows directly from (1.7).

\( c \Rightarrow a \). First, we shall prove that the set \( G = \{\delta_m(- + x), m \in \mathbb{N}, x \in \mathbb{R}^n\} \) is dense in \( \mathcal{F} \). Suppose that \( T \in \mathcal{F}^r \) and that

$$
\langle T, \delta_m(- + x) \rangle = 0, m \in \mathbb{N}, x \in \mathbb{R}^n.
$$

It follows that \( (T \ast \delta_m)(-x) = 0, m \in \mathbb{N}, x \in \mathbb{R}^n \). Then, for any \( \varphi \in \mathcal{F}, \langle T \ast \delta_m, \varphi \rangle = 0, m \in \mathbb{N} \), and consequently

$$
\langle T, \varphi \rangle = \lim_{m \to \infty} \langle T, \delta_m \ast \varphi \rangle = \lim_{m \to \infty} \langle T \ast \delta_m, \varphi \rangle = 0.
$$
This implies that $T = 0$ and hence, the set $G$ is dense in $\mathcal{D}$ by the Hahn-Banach theorem. Thus, by (1.10) and the Banach–Steinhaus theorem, c) implies a).

a) $\Rightarrow$ d). Note that (1.1) implies the strong convergence of $T(h)/c(h)$ to $U$ in $\mathcal{D}'$ and $\mathcal{D}'^*$ respectively. Since the convolution in $\mathcal{D}'$ and $\mathcal{D}'^*$ is hypocontinuous, it follows the following equality in the sense of convergence in $\mathcal{E}(\mathbb{R}^n)$

$$\lim_{h \in \Gamma, \|h\| \to \infty} \left( \frac{T(\cdot + h)}{c(h)} * \varphi \right) = \left( \lim_{h \in \Gamma, \|h\| \to \infty} \frac{T(\cdot + h)}{c(h)} \right) * \varphi = U * \varphi$$

in both cases ($\varphi \in \mathcal{D}$, respectively $\varphi \in \mathcal{D}'$).

The paper which can be consulted for related results is [128].

1.4 Basic properties of the $S$-asymptotics

**Theorem 1.2.** Let $T \in \mathcal{F}_g$.

a) If $T(x + h) \sim c(h)U(x)$, $h \in \Gamma$, then for every $k \in \mathbb{N}_0$, $T^{(k)}(x + h) \sim c(h)U^{(k)}(x)$, $h \in \Gamma$.

b) Assume additionally that $\mathcal{F}_g$ is a Montel space. Let $g \in M_{\{1\}}$ (set of multipliers of $\mathcal{F}_g$ (cf. 0.6)); let $c, c_1$ be positive functions. If for every $\varphi \in \mathcal{F}_g$, $(g(x + h)/c_1(h))\varphi(x)$ converges to $G(x)\varphi(x)$ in $\mathcal{F}_g$ when $h \in \Gamma$, $\|h\| \to \infty$ and if $T(x + h) \sim c(h)U(x)$, $h \in \Gamma$, then $g(x + h)T(x + h) \sim c_1(h)c(h)G(x)U(x)$, $h \in \Gamma$.

c) If $T \in \mathcal{F}_g$ and supp $T$ is compact, then $T(x + h) \sim c(h) \cdot 0$, $h \in \Gamma$, for every positive function $c$.

d) Suppose that $\mathcal{F}_g$ is a Montel spaces in which the convolution with compactly supported elements is well defined and hypocontinuous (cf. 0.6). Let $S \in \mathcal{F}_g$, supp $S$ being compact. If $T(x + h) \sim c(h)U(x)$, $h \in \Gamma$, then $(S \ast T)(x + h) \sim c(h)(S \ast U)(x)$, $h \in \Gamma$.

e) Let $T \in \mathcal{F} = \mathcal{D}'^*$ and $T(x + h) \sim c(h) \cdot U(x)$, $h \in \Gamma$, in $\mathcal{D}'^*$. Assume that (M.2) holds. Let $P(D)$ be an ultradifferential operator of class *. Then

$$(P(D)T)(x + h) \sim c(h) \cdot (P(D)U)(x)$$

$h \in \Gamma$, in $\mathcal{D}'^*$.

f) Let $T \in \mathcal{F}' = Q(\mathcal{D}^*)$. Let $P(D)$ be a local operator and let $T(x + h) \sim c(h) \cdot U(x)$, $h \in \Gamma$, $\|h\| \to \infty$ in $Q(\mathcal{D}^*)$, then $(P(D)T)(x + h) \sim c(h) \cdot (P(D)U)(x)$, $h \in \Gamma$, $\|h\| \to \infty$ in $Q(\mathcal{D}^*)$, as well.
Proof. a) The assertion is a consequence of the definition of the derivative of a generalized function. Namely,

\[
\lim_{h \in \Gamma, \|h\| \to \infty} \left \langle T^{(k)}(x + h) c(h), \varphi(x) \right \rangle = \lim_{h \in \Gamma, \|h\| \to \infty} \left \langle \frac{T(x + h) c(h)}{c(h)}, (-1)^k \varphi^{(k)}(x) \right \rangle \\
= (-1)^k \langle U(x), \varphi^{(k)}(x) \rangle \\
= \langle U^{(k)}, \varphi \rangle, \varphi \in \mathcal{F}_g.
\]

b) Since \( \mathcal{F}_g \) is a Montel space, we have that \( \lim_{h \in \Gamma, \|h\| \to \infty} \frac{T(x + h)}{c(h)} = U(x) \) in \( \mathcal{F}_g \) with respect to strong topology. As \( \frac{g(x + h)}{c_1(h)} - G(x) \varphi(x), h \in \Gamma, \|h\| \geq 0, \) is a bounded set in \( \mathcal{F}_g \), it follows:

\[
\lim_{h \in \Gamma, \|h\| \to \infty} \langle g(x + h)T(x + h)(c_1(h)c(h)), \varphi(x) \rangle \\
= \lim_{h \in \Gamma, \|h\| \to \infty} \left \langle T(x + h)/c(h), \left( \frac{g(x + h)}{c_1(h)} - G(x) \right) \varphi(x) \right \rangle \\
+ \lim_{h \in \Gamma, \|h\| \to \infty} \langle T(x + h)/c(h), G(x) \varphi(x) \rangle \\
= \left \langle U(x), \lim_{h \in \Gamma, \|h\| \to \infty} \left( \frac{g(x + h)}{c_1(h)} - G(x) \right) \varphi(x) \right \rangle + \langle U(x), G(x) \varphi(x) \rangle \\
= 0 + \langle U(x)G(x), \varphi(x) \rangle = \langle UG, \varphi \rangle, \varphi \in \mathcal{F}_g.
\]

c) For each \( \varphi \in \mathcal{F}_0 \) there exists \( r_\varphi > 0 \) such that \( \text{supp} \varphi \subset B(0, r_\varphi) = \{ x \in \mathbb{R}^n; \|x\| < r_\varphi \} \). The support of \( T(x + h) \) is \( (\text{supp} \, T \setminus h) \). Thus, by our assumption, there exists \( \beta_{r_\varphi} \) such that for all \( h \in \Gamma, \|h\| > \beta_{r_\varphi} \) the set \( (\text{supp} \, T \setminus h) \cap B(0, r_\varphi) \) is empty and consequently \( \langle T(x + h), \varphi(x) \rangle = 0, h \in \Gamma, \|h\| \geq \beta_{r_\varphi} \).

d) By definition of the convolution

\[
\langle (S \ast T)(x + h), \varphi(x) \rangle = \langle (S \ast T)(x), \varphi(x - h) \rangle \\
= \langle S_t \times T_y, \alpha(t) \varphi(t + y - h) \rangle \\
= \langle S_t \times T_y(\cdot + h), \alpha(t) \varphi(\cdot + y) \rangle.
\] (1.12)

Hence, \( (S \ast T)(x + h) = (S \ast T(\cdot + h))(x) \).
Asymptotic Behavior of Generalized Functions

Since in a Montel space the weak and the strong convergence are equivalent, we have by (1.12)

\[ \lim_{h \in \Gamma, \|h\| \to \infty} (S \ast T)(x+h)/c(h) = \lim_{h \in \Gamma, \|h\| \to \infty} \left( \frac{S \ast \frac{T(\cdot+h)}{c(\cdot)}}{c(h)} \right)(x) = (S \ast U)(x). \]

For the proof of e) and f), we note that an ultradifferential operator \( P(D) \) maps continuously \( D' \ast \) into \( D' \ast \) and a local operator maps continuously \( Q(D') \) into \( Q(D') \).

\( \square \)

Remark. From assertion a) of Theorem 1.2, a natural question arises for spaces \( \mathcal{F}' \): The limit \( U \) can be a constant generalized function, hence \( U' = 0 \). Is there a positive function \( \tilde{c} \) such that \( T' \) has S-asymptotics related to this \( \tilde{c} \), but with a limit different from zero?

In general the answer is negative as shown by the following example: Let \( T \) be defined by \( x^2 + \sin(\exp(x^2)) \), \( x \in \mathbb{R} \). Then, \( T(x+h) \sim h^2 \cdot 1, \ h \in \mathbb{R}_+ \). But \( T'(x) = 2x(1 + \exp(x^2) \cos(\exp(x^2))) \). The same situation is obtained with the distribution \( f(x) = x^2 + x \sin x \).

We can now formulate an open problem: Suppose that \( T(x+h) \sim c(h)U(x), \ h \in \Gamma \). If \( U(x) \) is a constant generalized function, then the problem is to find some additional conditions on \( T \) which guarantee the existence of a function \( \tilde{c} \) such that \( T' \) has the S-asymptotics in \( \Gamma \) related to \( \tilde{c} \).

More generally, let \( S \in \mathcal{F}' \) and \( T = (\partial/\partial x_k)S \). If \( T(x+h) \sim c(h)U(x), \ h \in \Gamma \). The question is what we can say about the S-asymptotics of \( S \).

Recall the well known result: Let \( h \) be a real-valued function which has the first derivative \( h'(x) \neq 0 \), \( x \geq x_0 \) and \( h(x) \to \infty, \ x \to \infty \). If a function \( F \) has its first derivative on \( (x_0, \infty) \) such that there exists \( \lim_{x \to \infty} F'(x)/h'(x) = A \), then there exists \( \lim_{x \to \infty} F(x)/h(x) = A \), as well.

We know that an opposite assertion does not hold, and this is at the basis of the open problem quoted above.

We give a theorem to illustrate the relation between the S-asymptotics of a distribution and the S-asymptotics of its primitive. We refer to [114] for the proof.

**Theorem 1.3.** 1) Let \( f, g \in \mathcal{D}'(\mathbb{R}) \) and for some \( m \in \mathbb{N} \), \( g^{(m)} = f \).
1. S-asymptotics in $\mathcal{F}_g$

a) If $f(x+h) \sim h^n L(h) \cdot 1$, $h \in \mathbb{R}_+$, where $\nu > -1$, then $g(x+h) \sim h^{\nu + m} L(h) \cdot 1$, $h \in \mathbb{R}_+$.

b) If $f(x+h) \sim \exp(ah)L(\exp h) \exp(\alpha x)$, $h \in \mathbb{R}_+$, $\alpha \in \mathbb{R}$, and

$$\int_0^x \exp(ah)L(\exp h)dh \to \infty, \text{ when } x \to \infty, \text{ then}$$

$$g(x+h) \sim \left( \int_0^{h_{m-1}} \int_0^{h_1} \ldots \int_0^{h_1} \exp(\alpha t)L(\exp t)dt dh_1 \ldots dh_{m-1} \right) \exp(\alpha x), h \in \mathbb{R}_+.$$

2) Let $\phi_0 \in \mathcal{D}^p(\mathbb{R})$ such that $\int \phi_0(t)dt = 1$. If

$$\lim_{h \to \infty} \left( \frac{g(t)(x+h)}{\exp(ah)L(\exp h)} , \phi_0(x) \right) = \alpha^i(\exp(\alpha x), \phi_0(x)), i = 0, 1, \ldots, m - 1$$

and

$$f(x+h) \sim \exp(ah)L(\exp h)\alpha^m \exp(\alpha x), h \in \mathbb{R}_+,$$

then

$$g(x+h) \sim \exp(ah)L(\exp h)\exp(\alpha x), h \in \mathbb{R}_+.$$

3) Suppose that $T \in \mathcal{D}'$, $\Gamma = \{x \in \mathbb{R}^n; x = (0, \ldots, x_k, 0, \ldots, 0)\}$ and $T = (\partial/\partial x_k)S$. If $T(x+h) \sim c(h)U(x)$, $h \in \Gamma$ and $c(h)$ is locally integrable in $h_k$ such that

$$c_1(h_k) = \int_{h_k}^h c(v)dv_k \to \infty \text{ as } h_k \to \infty, h_k^0 \geq 0,$$

then $S(x+h) \sim c_1(h)U(x)$, $h \in \Gamma$.

4) Suppose that $S \in \mathcal{D}'$ and that for an $m \in \{1, 2, \ldots, n\},$

$$(D_{x_m}S)(x+h) \sim c(h) \cdot U(x), \quad h \in \Gamma.$$ Let $V \in \mathcal{D}'$, $D_{x_m}V = U$ and $\phi_0 \in \mathcal{D}(\mathbb{R})$, $\int \phi_0(\tau)d\tau = 1$. Let

$$\lim_{h \in \Gamma, \|h\| \to \infty} \langle S(x+h)/c(h), \phi_0(x_m)\lambda_m(\tilde{x}) \rangle = \langle V, \phi_0\lambda_m \rangle,$$

where $\tilde{x} = (x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n)$ and

$$\lambda_m(\tilde{x}) = \int_{\mathbb{R}^{m}} \psi(x_1, \ldots, x_m, \ldots, x_n)dx_m, \quad \psi \in \mathcal{D}.$$ Then $S(x+h) \sim c(h)V(x), h \in \Gamma.$
The following theorem asserts that the S-asymptotics is a local property if the elements of $\mathcal{F}_g$ are compactly supported.

**Theorem 1.4.** Let $T_1, T_2 \in \mathcal{F}_0$. Let the open set $\Omega \subset \mathbb{R}^n$ have the following property: for every $r > 0$ there exists a $\beta_r > 0$ such that the ball $B(0, r) = \{x \in \mathbb{R}^n; \|x\| < r\}$ is in $\{\Omega - h; h \in \Gamma, \|h\| \geq \beta_r\}$. If $T_1 = T_2$ on $\Omega$ and $T_1(x + h) \sim c(h)U(x)$, $h \in \Gamma$, then $T_2(x + h) \sim c(h)U(x)$, $h \in \Gamma$, as well.

**Proof.** Let $\varphi \in \mathcal{F}_0$ with $\text{supp} \varphi \subset B(0, r)$. We shall prove that

$$\lim_{h \to \infty} \left( \frac{T_1(x + h) - T_2(x + h)}{c(h)} \right) \varphi(x) = 0.$$  \hspace{1cm} (1.13)

The complement of the set $\text{supp}(T_1(x + h) - T_2(x + h))$ contains the set $\{\Omega - h, h \in \Gamma\}$. By our supposition the number $\beta_r$ is fixed in such a way that the sets $\{\Omega - h; h \in \Gamma, \|h\| \geq \beta_r\}$ contain $B(0, r)$ and consequently $\text{supp} \varphi$. Since $T_1$ has the S-asymptotics related to $c$ and with the limit $U$, (1.13) implies

$$\lim_{h \to \infty} \left( \frac{T_2(x + h)}{c(h)} \right) \varphi(x) = \lim_{h \to \infty} \left( \frac{T_1(x + h)}{c(h)} \right) \varphi(x) = \langle U, \varphi \rangle.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} ∎

**Remark.** The open set $\Omega$, by its property, has to contain a set $\Gamma^0 \cap \{x \in \mathbb{R}^n; \|x\| > R\}$ where $\Gamma^0$ is an open acute cone such that $\Gamma \subset \Gamma^0$ and $R$ is a positive number.

### 1.5 S-asymptotic behavior of some special classes of generalized functions

#### 1.5.1 Examples with regular distributions

1. $\exp(a \cdot (x + h)) \overset{\sim}{\sim} \exp(a \cdot h) \exp(a \cdot x)$, $h \in \mathbb{R}^n$.

2. $\exp(\sqrt{(x + h)^2 + (x + h)}) \overset{\sim}{\sim} \exp h \exp \left( x + \frac{1}{2} \right)$, $h \in \mathbb{R}_+$.

3. Let $w \in S^{n-1} = \{x \in \mathbb{R}^n; \|x\| = 1\}$, $p = (p_1, \ldots, p_n) \in \mathbb{R}_+^n$ and $\Gamma = \{qw; q \in \mathbb{R}_+\}$. Denote, $J = \{k \in \{1, \ldots, n\}; w_k \neq 0\}$ and $\beta = \sum_{i \in J} p_i$. Then

$$(x + h)^p \overset{\sim}{\sim} q^\beta \prod_{i \in J} w_i^{p_i} \cdot 1, \quad h \in \Gamma,$$
1. S-asymptotics in $F'$

and

$$(1 + x + h)^{-p} \sim q_{\beta}^{-p} \prod_{i \in J} w_i^{a_i} \cdot 1, \ h \in \Gamma,$$

$$((x + h)^{p} = (x_1 + h_1)^{a_1} \ldots (x_n + h_n)^{a_n}).$$

4. For a slowly varying function $L(t), t \geq \alpha > 0$ we have

$L(t + h) \sim L(h) \cdot 1, \ h \in \mathbb{R}_+.$

Namely,

$$\lim_{h \to \infty} \langle L(t + h)/L(h), \varphi(t) \rangle = \lim_{h \to \infty} \int \varphi(t)L(t + h)/L(h)dt$$

$$= \lim_{h \to \infty} \int e^{-r} \varphi(y)\log(y/L)\log(q)/\log(h)dy = \int \varphi(t)dt, \ \varphi \in \mathcal{D}(\mathbb{R}).$$

We used above that $L(\log y)$ is also a slowly varying function (cf. 0.3) and that $L(\log(uh))/L(\log h)$ converges uniformly to 1 as $h \to \infty$ if $u$ stays in compact intervals $[\alpha_1, \alpha_2], \ 0 < \alpha_1 < \alpha_2 < \infty.$

5. Let $g \in L^1(\mathbb{R}).$ A distribution defined by $g$ has the S-asymptotic behavior related to $c = 1$ and with limit $U = 0.$ To show this, let $\alpha \in \mathbb{R}$ and

$$T(t) = \int_{\alpha}^{t} g(x)dx, \ t \in \mathbb{R}.$$ 

Then, $T(t + h) \sim 1 \cdot \int_{\alpha}^{\infty} g(x)dx, h \in \mathbb{R}_+.$ By Theorem 1.2 a), $g(t + h) \sim 1 \cdot 0, h \in \mathbb{R}_+.$

1.5.2 Examples with distributions in subspaces of $\mathcal{D}'$

6. Let $f_\alpha, \ \alpha \in \mathbb{R}$, be given as in 0.4. It defines a distribution in $\mathcal{D}'_{[0, \infty)}.$

For $\alpha = -k, \ k = 0, 1, \ldots, f_{-k} = \delta^{(k)}.$ Since $\delta^{(k)}$ is supported by $\{0\},$ it has the S-asymptotics zero related to every $c > 0$ (cf. Theorem 1.2 c)). For $\alpha > 0,$

$$f_\alpha(x + h) \sim (1/\Gamma(\alpha))h^{\alpha-1} \cdot 1, \ h \in \mathbb{R}_+.$$
Asymptotic Behavior of Generalized Functions

In the case \( \alpha < 0, \alpha \neq -1, -2, \ldots \), \( f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \cdot \text{PF}(H(x)x^{\alpha-1}) \), where \( \text{PF} \) is the finite part or “partie finie” (see [146], pp. 41–43).

Note, supp \( \phi(x - h) = \text{supp} \phi + h, \phi \in \mathcal{D}(\mathbb{R}) \). Thus for \( \phi \in \mathcal{D}(\mathbb{R}) \), we can find \( h_0 \) such that supp \( \phi(x - h) \subset (a, \infty), a > 0, h \geq h_0 \). Then

\[
\lim_{h \to \infty} \left( \frac{f_\alpha(x)}{h^{\alpha-1}}, \phi(x - h) \right) = \lim_{h \to \infty} \int_a^{\infty} \frac{1}{\Gamma(\alpha)} \left( \frac{x}{h} \right)^{\alpha-1} \phi(x - h)dx = \lim_{h \to \infty} \int_\mathbb{R} \frac{1}{\Gamma(\alpha)} \left( \frac{u + h}{h} \right)^{\alpha-1} \phi(u)du = \int_\mathbb{R} \frac{1}{\Gamma(\alpha)} \phi(u)du.
\]

Hence, \( f_\alpha(x + h) \sim h^{\alpha-1} \cdot \frac{1}{\Gamma(\alpha)}, h \in \mathbb{R}_+, \alpha < 0, \alpha \neq -1, -2, \ldots \)

7. If \( T \in \mathcal{S}' \), then there exists a real number \( k_0 \) such that \( T \) has \( S \)-asymptotic behavior related to \( c(h)\|h\|^{k_0} \), where \( c(h) \) tends to infinity as \( |h| \to \infty, h \in \mathbb{R}^n \) and with limit \( U = 0 \).

By Theorem VI, chapter VII in [146], there exists a number \( k_0 \) such that the set of distributions \( \{ T(x + h)/(1 + \|h\|^2)^{k_0/2}; h \in \mathbb{R}^n \} \) is bounded in \( \mathcal{D}' \). Hence, this set is weakly bounded and

\[
\left\langle T(x + h)/(c(h)\|h\|^{k_0}), \varphi(x) \right\rangle = \left\langle \frac{1 + \|h\|^2}{(1 + \|h\|^2)^{k_0/2}} \frac{T(x + h)}{c(h)\|h\|^{k_0}}, \varphi(x) \right\rangle
\]

tends to zero as \( \|h\| \to \infty \).

8. Let \( T \in \mathcal{D}'(\mathbb{R}) \) have the following property:

For a \( \delta \)-sequence \( \langle \delta_m \rangle \) there is a sequence \( \langle p_m \rangle \) in \( \mathbb{R} \), such that \( p_m \to p \neq 0, m \to \infty \), and

\[
\lim_{h \to \infty} \frac{(T * \delta_m)(h)}{c(h)} = p_m, \quad m \in \mathbb{N},
\]

where the limit is uniform for \( m \in \mathbb{N} \).

Then \( T \) has \( S \)-asymptotics related to \( c \).

We will prove that b) in Theorem 1.1 is satisfied, which is equivalent to a) in the same theorem. For every compact set \( K \subset \mathbb{R} \)

\[
\frac{(T * \delta_m)(x + h)}{c(h)} = \frac{(T * \delta_m)(x + h)}{c(x + h)} \frac{c(x + h)}{c(h)} \to p_m \exp(\alpha x), h \in \Gamma, \|h\| \to \infty,
\]

uniformly for \( x \in K \), because of \( \frac{c(x + h)}{c(h)} \to \exp(\alpha x), \|h\| \to \infty \), uniformly for \( x \in K \) (cf. Remarks after Proposition 1.2). Then, \( T \) has the \( S \)-asymptotics related to \( c \).
1. S-asymptotics in $F_g$

Property (*) is not equivalent to the existence of the S-asymptotics. The next example illustrates that this condition is not necessary.

Assume that $S \in C(R) \cap L^1(R)$ but not being bounded on $R$. This function has S-asymptotics equal to zero related to $c = 1$ (see 5.). For $T$ we take $1 + S(x)$. Then, $T(x + h) \sim 1 \cdot 1$, $h \in R_+$ and

$$\lim_{h \to \infty} [(1 + S) \ast \delta_m](h) = \lim_{h \to \infty} \langle 1 + S(x + h), \delta_m(x) \rangle = \langle 1, \delta_m(x) \rangle = p_m.$$  

This limit is not uniform in $m \in N$ because $\lim_{m \to \infty} [(1 + S) \ast \delta_m](h)$ does not necessarily exist.

9. Every distribution in $D'_L$, $1 \leq p < \infty$ has S-asymptotic behavior related to $c = 1$, and $\Gamma = R^n$, with limit $U = 0$. Let us show this.

By Theorem XXV, Chapter VI in [146] it follows that $(T \ast \varphi) \in L^p(R^n)$ for every $\varphi \in D$. Every derivative of $(T \ast \varphi)$, $\frac{\partial}{\partial h_k}(T \ast \varphi)(h) = (T \ast \frac{\partial}{\partial x_k} \varphi)(h)$, $h \in R^n$, is also in $L^p(R^n)$. Hence $(T \ast \varphi) \in D_L$. We know that every element of $D_L$, $1 \leq p < \infty$ is bounded over $R^n$ and tends to zero when $\|h\| \to \infty$ ([146], p. 199).

It has been examined in [153] how slowly the function $(T \ast \varphi) \in D_L$, $\varphi \in D$, tends to zero as $\|h\| \to \infty$.

In fact, the question is:

Let $p \geq 1$. Is it possible to find a positive function $c$ such that $c(x) \to 0$, $\|x\| \to \infty$ and $|\varphi(x)/c(x)| \leq C_\varphi$, $\|x\| \geq R_\varphi$, for every $\varphi \in D_L$? $C_\varphi$ and $R_\varphi$ are positive constants depending on $\varphi$.

The answer is negative (see [153]).

A similar question can be asked for $T \in D'_L$, $1 \leq p < \infty$, but related to the S-asymptotics. Precisely, whether there exists $c(h) > 0$, $c(h) \to 0$, $\|h\| \to \infty$, such that

$$|(T(x + h)/c(h), \varphi(x))| \leq M_\varphi, \|h\| > \beta_\varphi \quad \varphi \in D,$$

where $\beta_\varphi$ and $M_\varphi$ are positive constants depending on $\varphi$.

The answer to this question is also negative (cf. [153]).

10. Let $T \in K'_p$. Then there exists $k_0 \in N_0$ such that $T$ has S-asymptotic behavior with limit $U = 0$ related to $c(h) \exp(k_0\|h\|^p)$, where $c(h)$ tends to infinity as $\|h\| \to \infty$.
First, we prove that there exists a positive integer \( k \), such that the set
\[
\{ T(\cdot + h) \exp(-k\|h\|^p), \ h \in \mathbb{R}^n \}
\]
is bounded in \( \mathcal{D}' \).

We start by giving a bound for the seminorms \( \nu_k(\varphi(\cdot - h)) \), \( \varphi \in \mathcal{K}_p \):

\[
\nu_k[\varphi(\cdot - h)] = \sup_{x \in \mathbb{R}^n \cdot |a| \leq k} \exp(k\|x\|^p)|D^a \varphi(x - h)|
\]
\[
= \sup_{x \in \mathbb{R}^n \cdot |a| \leq k} \exp(k\|x + h\|^p)|D^a \varphi(x)|
\]
\[
\leq \exp(2p\|k\|p) \sup_{x \in \mathbb{R}^n \cdot |a| \leq 2pk} \exp(2p\|k\|p)|D^a \varphi(x)|
\]
\[
\leq \exp(2p\|k\|p)\nu_{2pk}(\varphi).
\]

By assumption, \( T \) is a continuous linear functional on \( \mathcal{K}_p \). Note, the sequence of norms \( \langle \nu_k \rangle_k \) is increasing. Thus, there exist \( \varepsilon > 0 \) and \( k_0 \in \mathbb{N}_0 \) such that

\[
|\langle T, \varphi \rangle| \leq 1 \quad \text{for} \quad \varphi \in \mathcal{K}_p, \ \nu_{k_0}(\varphi) \leq \varepsilon.
\]

This inequality holds for all \( k \geq k_0 \). Hence

\[
|\langle T, \varphi \rangle| \leq \varepsilon^{-1}\nu_k(\varphi), \ k \geq k_0 \quad \text{for every} \quad \varphi \in \mathcal{K}_p.
\]

We know that \( \mathcal{D} \subset \mathcal{K}_p \) and that the inclusion is continuous. Let \( \varphi \in \mathcal{D} \), Then

\[
|\langle \exp(-2pk\|h\|^p)T(x + h), \varphi(x) \rangle| = |\langle T(x), \exp(-2pk\|h\|^p)\varphi(x - h) \rangle|
\]
\[
\leq \varepsilon^{-1}\exp(-2pk\|h\|^p)\nu_k[\varphi(x - h)] \leq \varepsilon^{-1}\nu_{2pk}(\varphi), \ k > k_0.
\]

We can choose \( k_0 \geq 2pk \). The set \( \{ \exp(-k_0\|h\|^p)T(x + h); h \in \mathbb{R}^n \} \) is bounded in \( \mathcal{D}' \) (and weakly bounded in \( \mathcal{D}' \), as well).

Now, for every \( \varphi \in \mathcal{D} \):

\[
\lim_{\|h\| \to \infty} \langle \exp(-k_0\|h\|^p)T(x + h)/c(h), \varphi(x) \rangle
\]
\[
= \lim_{\|h\| \to \infty} \frac{1}{c(h)}\langle \exp(-k_0\|h\|^p)T(x + h), \varphi(x) \rangle = 0.
\]
1. S-asymptotics in $F'_g$

1.5.3 S-asymptotics of ultradistributions and Fourier hyperfunctions — Comparisons with the S-asymptotics of distributions

11. If a distribution has S-asymptotics in $D'$ (see Definition 1.1), it has the same S-asymptotics in $D'\ast$, as well. But the opposite is not true. This is illustrated by the following example:

$$T(x) = 1 + \sum_{n=1}^{\infty} \delta^{(n)}(x-n)/M_n, \ x \in \mathbb{R},$$

has S-asymptotics in $D'(M_p)(\mathbb{R})$, but it does not have S-asymptotics in $D'(\mathbb{R})$. We will show this. Let $\phi \in D(M_p)(\mathbb{R})$.

$$\left\langle \sum_{n=1}^{\infty} \delta^{(n)}(x+h-n)/M_n, \phi(x) \right\rangle = \sum_{n=1}^{\infty} (-1)^n \phi^{(n)}(n-h)/M_n \to 0, \ h \to \infty.$$  

This is a consequence of the property

$$\sup_{x \in \mathbb{R}} |\phi^{(n)}(x)|/k^n M_n \to 0, \ n \to \infty \text{ for every } k > 0.$$  

Suppose that there exists a function $c$ such that for every $\varphi \in D(\mathbb{R})$

$$\sum_{n=1}^{\infty} (-1)^n \varphi^{(n)}(n-h)/M_n c(h),$$

converges, as $h \to \infty$. Taking $h = n$ this implies that $\varphi^{(n)}(0)/M_n c(n)$ converges to zero, as $n \to \infty$, for every $\varphi \in D(\mathbb{R})$. However, such a function $c$ does not exist (Borel’s theorem). Namely, for every given sequence $M_n c(n), \ n \in \mathbb{N}$, there exists $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi^{(n)}(0)/M_n c(n) \to \infty$, as $n \to \infty$.

This example is the motivation for the following assertion.

12. Let $T \in D'(\mathbb{R})$ be such that $\lim_{h \to \infty} T(x+h)/c(h)$ exists in $D'\ast(\mathbb{R})$. Assume that for some $s \in \mathbb{N}$ and $\omega \in D(\mathbb{R})$ with the property

$$\int_{\mathbb{R}} \omega(t) dt = 1, \int_{\mathbb{R}} t^j \omega(t) dt = 0, \ j = 1, \ldots, s,$$

the following limit

$$\lim_{h \to \infty} \left\langle \frac{T(t+h)}{c(h)}, \frac{t^p}{p^{j+1}} \omega \left( \frac{t}{p} \right) \right\rangle, \ p \in (0,1],$$

exists uniformly in $p$. Then, $\lim_{h \to \infty} T(x+h)/c(h)$ exists in $D'(\mathbb{R})$ as well.
Set \( p \in \mathcal{D}(\mathbb{R}) \), \( h \geq h_0 > 0 \). The function \( F_h(t) = \langle T(x + h + t), \phi(x) \rangle \), \( t \in \mathbb{R} \), is smooth and by the Taylor formula, we have:

\[
F_h(t) = F_h(0) + t \frac{d}{dt} F_h(0) + \cdots + \frac{t^{s-1} \omega}{(s-1)!} \frac{d^{s-1} F_h}{dt^{s-1}}(0)
\]

\[
+ \frac{t^s}{(s-1)!} \int_0^1 (1 - p)^{s-1} F_h^{(s)}(pt) dp, \quad s \geq 1, \quad 0 < p \leq 1.
\]

This implies

\[
\langle T(x + h), \phi(x) \rangle = \langle T(x + h + t), \phi(x) \rangle - \langle T(x + h), \phi'(x) \rangle - \cdots - \frac{(-1)^{s-1} t^{s-1}}{(s-1)!} \langle T(x + h), \phi^{(s-1)}(x) \rangle
\]

\[
- \frac{(-1)^s t^s}{(s-1)!} \int_0^1 (1 - p)^{s-1} \langle T(x + h + pt), \phi^{(s)}(x) \rangle dp,
\]

\( h \geq h_0, \quad t \in \mathbb{R} \).

Multiplying both sides of the last equality by \( \omega(t) \), integrating with respect to \( t \), and using Fubinni’s theorem, we obtain

\[
\langle T(x + h), \phi(x) \rangle = \langle \langle T(x + h + t), \omega(t) \rangle, \phi(x) \rangle - \cdots - \frac{(-1)^s}{(s-1)!} \int_0^1 (1 - p)^{s-1} \langle \langle T(x + h + pt), t^s \omega(t) \rangle, \phi^{(s)}(x) \rangle dp.
\]

Set

\[
G(x, h, p) = \langle T(x + h + pt), t^s \omega(t) \rangle, \quad x \in \mathbb{R}, \quad h \geq h_0, \quad p \in [0, 1].
\]

We have \( G(x, h, 0) = 0 \), \( x \in \mathbb{R}, \quad h \geq h_0 \) and

\[
\lim_{h \to \infty} G(x, h, p)/c(h) = Ce^{ax} \langle \langle e^{ap}, t^s \omega(t) \rangle, \quad x \in \mathbb{R}, \quad p \in (0, 1],
\]

for some \( a \in \mathbb{R} \) (cf. Proposition 1.2 a)), where the limit is uniform in \( p \in (0, 1] \) and \( x \in \text{supp} \phi \). The limit function is continuous in \( x \in \mathbb{R} \) and \( p \in (0, 1] \), because of \( \langle \langle e^{ap}, t^s \omega(t) \rangle \to 0 \), as \( p \to 0 \). Because of that, we obtain

\[
\lim_{h \to \infty} \frac{1}{c(h)} \langle \langle T(x + h + pt), t^s \omega(t) \rangle, \phi^{(s)}(x) \rangle
\]

\[
= \left\langle \lim_{h \to \infty} \frac{1}{c(h)} \langle T(x + h + pt), t^s \omega(t) \rangle, \phi^{(s)}(x) \right\rangle
\]

\[
= \langle Ce^{ax + apt}, t^s \omega(t) \rangle, \phi^{(s)}(x) \rangle = Ce^{ax} \langle \phi^{(s)}(x) \rangle \langle \langle e^{ap}, t^s \omega(t) \rangle.
\]
This implies
\[
\lim_{h \to \infty} \left\langle \frac{1}{c(h)} T(x+h), \phi(x) \right\rangle = \left\langle \left\langle \lim_{h \to \infty} \frac{T(x+h+t)}{c(h)}, \omega(t) \right\rangle, \phi(x) \right\rangle \\
- \cdots - \frac{(-1)^s}{(s-1)!} \int_0^1 (1-p)^{s-1} (e^{ap t}, t^s \omega(t)) dp (Ce^{ax}, \phi(s)(x)),
\]
which proves the assertion. \(\square\)

Such an assertion can be proved in the multi-dimensional case by adjusting the previous argument. But an open problem is to find necessary and sufficient conditions for a distribution \(T\), which has S-asymptotics in \(D'\), to have the S-asymptotics in \(D'\) as well.

13. We shall construct an ultradistribution out of the space of Schwartz distributions and having S-asymptotics.

Assume (M.2) holds. Let \(P(D)\) be an ultradifferential operator of class * of infinite order \((a_\alpha \neq 0\) for infinitely \(\alpha)\) (see 0.5.2). Then, \(P(D)\delta\) is an element of \(D'*\) which is not a distribution and which has S-asymptotics in \(D'*\) equal to zero related to any \(c\).

If \(T \in D'\) and \(T(x+h) \sim 1 \cdot 1, h \in \Gamma\) in \(D'\), the ultradistribution \(T + P(D)\delta\) is not a distribution, but \((T + P(D)\delta)(x+h) \sim 1 \cdot 1, h \in \Gamma\) in \(D'*\):
\[
\lim_{h \in \Gamma, \|h\| \to \infty} \left( (T(x+h), \phi(x)) + (P(D)\delta(x+h), \phi(x)) \right)
= (1, \phi) + \lim_{h \in \Gamma, \|h\| \to \infty} \langle \delta(x+h), \sum (-1)^m a_m \phi^{(m)}(x) \rangle = (1, \phi), \quad \phi \in D^*.
\]

14. Let \(P(D)\) be a local operator \(\sum_{|\alpha| \geq 0} b_\alpha D^\alpha, b_\alpha \neq 0, |\alpha| \geq 0\) (see 0.5.3.). The Fourier hyperfunction \(f = 1 + P(D)\delta\) has S-asymptotics related to \(c = 1\) in any cone \(\Gamma\) and with limit \(U = 1\), but \(f\) is not a distribution. For the S-asymptotics of \(f\) it is enough to prove that
\[
\lim_{h \in \Gamma, \|h\| \to \infty} \langle P(D)\delta(x+h), \varphi(x) \rangle = 0, \quad \varphi \in \mathcal{P}_*.
\]
Since \(P(D)\) maps \(\mathcal{P}_*\) into \(\mathcal{P}_*\),
\[
\langle P(D)\delta(x+h), \varphi(x) \rangle = \langle \delta(x+h), P(-D)\varphi(x) \rangle = \psi(h),
\]
where \( \psi = P(-D) \varphi \). By the property of the elements of \( \mathcal{P} \), (see 0.5.3), we have \( \lim_{h \in \Gamma, \|h\| \to \infty} \psi(h) = 0 \), for every cone \( \Gamma \).

A hyperfunction \( g \) supported by the origin is uniquely expressible as \( g = \hat{P}(D) \delta \), where \( \hat{P}(D) \) is a local operator. In such a way, the above proof implies that every Fourier hyperfunction with support at \( \{0\} \) has S-asymptotics with limit equal zero.

Since \( P(D) \delta = \sum_{|\alpha| \geq 0} b_\alpha D^\alpha \delta \) is a distribution if and only if \( b_\alpha \neq 0 \) for a finite number of \( \alpha \), we note that \( 1 + P(D) \delta \) is not a distribution, but it has the S-asymptotics related to \( c = 1 \).

We can also find coefficients \( b_\alpha \) of a local operator \( P(D) \) such that \( f = 1 + P(D) \delta \) is not defined by an ultradistribution belonging to the Gevrey class \( D^s \) or \( \mathcal{D}^s \), \( s > 1 \) (see [81]). For the sake of simplicity, we shall consider the one-dimensional case. Choose \( P(D) \) such that the coefficients of \( P(D) \) are: \( b_n = (n!)^{-1+c_n} \), \( n \in \mathbb{N} \), where \( c_n = (\log \log n)^{-1} \). With these coefficients, \( P(D) \) is a local operator. Namely,

\[
\lim_{n \to \infty} \sqrt[n]{b_n n!} = \lim_{n \to \infty} (n!)^{-1 + c_n} = 0.
\]

Also, any ultradistribution in the Gevrey class \( s > 1 \), supported by \( \{0\} \), is of the form

\[
J(D) \delta = \sum_{n=0}^{\infty} a_n D^n \delta, \text{ necessarily with } |a_n| \leq C k^n /(n!)^s
\]

for some constants \( k \) and \( C \) (Beurling’s type) or for any \( k > 0 \) with a constant \( C \) (Roumieu’s type). But the coefficients \( b_n = (n!)^{-1+c_n} \) do not satisfy these conditions, therefore \( P(D) \delta \) cannot represent an ultradistribution. Namely, since \( c_n \to 0 \) when \( n \to \infty \), for any \( s > 1 \), there exists \( n_0 \) such that \( 1 + c_n < s, n \geq n_0 \). Thus,

\[
(n!)^{-1+c_n} > C k^n /(n!)^s, \quad n \geq n_0, \quad k > 0.
\]

Consequently, \( P(D) \delta \) does not represent an ultradistribution of Gevrey type.

On the other hand, if we suppose that \( g = P(D) \delta \) is an ultradistribution with support \( \{0\} \) in the Gevrey class \( s > 1 \), then, we would have an ultradifferential operator \( J_1(D) \) such that

\[
g = J_1(D) \delta = \sum_{n=0}^{\infty} e_n D^n \delta, \quad |e_n| \leq C k^n /(n!)^s.
\]
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But in this case $J_1(D)$ is also a local operator and $J_1(D) \neq P(D)$. This contradicts the fact that a hyperfunction with support at $\{0\}$ is given by a unique local operator.

1.6 S-asymptotics and the asymptotics of a function

We suppose in this subsection that elements of $\mathcal{F}$ are compactly supported and, as usual, that the topology in $\mathcal{F}$ is stronger than the topology in $\mathcal{E}$. Recall, we use the notation $\mathcal{F}_0$ in this case.

Every locally integrable function $f$ defines an element of $\mathcal{F}_0'$ (regular generalized functions).

We shall compare the asymptotic behavior of a locally integrable function $f$ and the S-asymptotic behavior of the generalized function generated by it.

A function $f$ has asymptotics at infinity if there exists a positive function $c$ such that $\lim_{x \to \infty} f(x)/c(x) = A \neq 0$, (in short $f(x) \sim Ac(x), \ x \to \infty$).

1. The following example points out that a continuous and $L^1$-integrable function can have S-asymptotics as a distribution without having an ordinary asymptotics. Suppose that $g \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ has the property that $g(n) = n, n \in \mathbb{N}$ and that it is equal to zero outside suitable small intervals $I_n \ni n, n \in \mathbb{N}$. Denote by $f(t) = e^t \int_0^t g(x)dx, \ t \in \mathbb{R}$. It is easy to see that

$$f(t + h) \sim e^h \cdot e^t \int_0^\infty g(x)dx, \ h \in \mathbb{R}_+.$$  

By Theorem 1.2 a) $f'(t)$ has S-asymptotics related to $e^h$ and with the same limit. But, in view of the properties of $g$, $f'(t) = f(t) + e^t g(t)$ has not the same asymptotics (in the ordinary sense). Moreover, $g$ can be chosen so that $f'$ has no asymptotics at all.

2. The following example shows that a function $f$ can have asymptotic behavior without having S-asymptotics with limit $U$ different from zero. An example is $x \mapsto \exp(x^2), x \in \mathbb{R}$. Suppose that $\exp(x^2)$ has S-asymptotics related to a $e^h > 0, h \in \mathbb{R}_+$ with a limit $U$ different from zero. By Proposition 1.2 c), $U$ has the form $U(x) = C \exp(ax), C > 0$. Then, for
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For every \( \varphi \in \mathcal{F}_0 \) such that \( \varphi > 0 \) we have
\[
\lim_{h \to \infty} \frac{1}{c(h)} \int \exp[(x + h + h_0)^2] \varphi(x) dx = e^{ah_0} \langle Ce^{ax}, \varphi(x) \rangle.
\]
Therefore,
\[
e^{ah_0} \langle U, \varphi \rangle = \exp(h_0^2) \lim_{h \to \infty} \frac{1}{c(h)} \int e^{(x+h)^2} e^{2h_0(x+h)} \varphi(x) dx
\]
\[
\geq \exp(h_0^2) \langle U, \varphi \rangle, \quad \text{for every } h_0 > 0.
\]
But this inequality is absurd. Consequently, \( \exp(x^2) \) cannot have such an S-asymptotic behavior.

One can prove a more general assertion.

**Proposition 1.3.** Let \( f \in L^1_{\text{loc}}(\mathbb{R}) \subset \mathcal{F}'_0(\mathbb{R}) \) have one of the four properties for \( \alpha > 1, \beta > 0, x \geq x_0, h > 0, M > 0 \) and \( N > 0 \):

a) \( f(x + h) \geq M \exp(\beta h^\alpha) f(x) \geq 0 \),

a') \( -f(x + h) \geq -M \exp(-\beta h^\alpha) f(x) \geq 0 \),

b) \( 0 \leq f(x + h) \leq N \exp(-\beta h^\alpha) f(x) \),

b') \( 0 \leq -f(x + h) \leq -N \exp(-\beta h^\alpha) f(x) \).

Then \( f \) cannot have S-asymptotics with limit \( U \neq 0 \), but the function \( f \) can have asymptotics.

For the proof see ([135], p. 89).

It is easy to show that for some classes of real functions \( f \) on \( \mathbb{R} \) the asymptotic behavior at infinity implies the S-asymptotics.

**Proposition 1.4.** a) Let \( c \) be a positive function and let \( T \in L^1_{\text{loc}}(\mathbb{R}^n) \).

Suppose that there exist locally integrable functions \( U(x) \) and \( V(x) \), \( x \in \mathbb{R}^n \), such that for every compact set \( K \subset \mathbb{R}^n \)
\[
|T(x + h)/c(h)| \leq V(x), \quad x \in K, \quad \|h\| > r_K,
\]
\[
\lim_{h \in \Gamma, \|h\| \to \infty} T(x + h)/c(h) = U(x), \quad x \in K.
\]

Then, \( T(x + h) \overset{s}{\sim} c(h)U(x) \), \( h \in \Gamma \) in \( \mathcal{F}'_0 \).

b) Let \( T \in L^1_{\text{loc}}(\mathbb{R}) \) have the ordinary asymptotic behavior
\[
T(x) \sim \exp(\alpha x)L(\exp x), \quad x \to \infty, \quad \alpha \in \mathbb{R},
\]
where \( L \) is a slowly varying function. Then,
\[
T(x + h) \overset{s}{\sim} \exp(\alpha h)L(\exp h) \exp(\alpha x), \quad h \in \mathbb{R}^+, \text{ in } \mathcal{F}'_0(\mathbb{R}).
\]
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Proof. a) For every $\varphi \in F'_0$

\[
\lim_{h \in \Gamma, \|h\| \to \infty} \left\langle \frac{T(x + h)}{c(h)}, \varphi(x) \right\rangle = \lim_{h \in \Gamma, \|h\| \to \infty} \int_{\mathbb{R}^n} \frac{T(x + h)}{c(h)} \varphi(x) dx. \quad (1.14)
\]

Since supp $\varphi \subset K \subset \mathbb{R}^n$ and $T$ has all the listed properties, Lebesgue’s theorem implies the result.

b) It is enough to use in (1.14) that $L(yt)/L(t) \to 1$, $t \to \infty$ uniformly in $y$, when $y$ stays in compact interval contained in $\mathbb{R}_+$.

A more general result is the following one ([135], pp. 89–90).

Proposition 1.5. Let $\Gamma$ be a cone and let $\Omega \subset \mathbb{R}^n$ be an open set such that for every $r > 0$ there exists a $\beta_r$ such that $B(0, r) \subset \{ \Omega - h; h \in \Gamma, \|h\| \geq \beta_r \}$.

Suppose that $G \in L^1_{\text{loc}}(\Omega)$ and it has the following properties: There exist locally integrable functions $U$ and $V$ in $\mathbb{R}^n$ such that for every $r > 0$ we have

\[ |G(x + h)/c(h)| \leq V(x), \quad x \in B(0, r), \quad h \in \Gamma, \quad \|h\| \geq \beta_r; \]

\[ \lim_{h \in \Gamma, \|h\| \to \infty} G(x + h)/c(h) = U(x), \quad x \in B(0, r). \]

If $G_0 \in F'_0$ coincides with $G$ on $\Omega$, then

\[ G_0(x + h) \overset{\ast}{\sim} c(h)U(x), \quad h \in \Gamma. \]

Proof. By Theorem 1.4, it is enough to prove that

\[ \lim_{h \in \Gamma, \|h\| \to \infty} \int_{\Omega} G(x + h)/c(h) \varphi(x) dx = \int_{\mathbb{R}^n} U(x)\varphi(x) dx; \]

but as in the proof of Proposition 1.4 a), the exchange of the limit and the integral sign is justified by our assumptions and Lebesgue’s theorem.

The following proposition gives a sufficient condition under which the S-asymptotics of $f \in L^1_{\text{loc}}(\mathbb{R})$, in $\mathcal{D}'(\mathbb{R})$, implies the ordinary asymptotic behavior of $f$.

Proposition 1.6. Let $f \in L^1_{\text{loc}}(\mathbb{R})$, $c(h) = h^\beta L(h)$, where $\beta > -1$, and $L$ be a slowly varying function. If for some $m \in \mathbb{N}, x^m f(x), x > 0$, is monotonous and $f(x + h) \overset{\ast}{\sim} c(h) \cdot 1, h \in \mathbb{R}_+$, in $\mathcal{D}'(\mathbb{R})$, then

\[ \lim_{h \to \infty} f(h)/c(h) = 1. \]

If we suppose that $L$ is monotonous, then we can omit the hypothesis $\beta > -1$.

For the proof see [115].
1.7 Characterization of the support of $T \in F'_0$

We suppose in this subsection that the topology in $F'_0$ is defined in such a way that a sequence $\{\varphi_n\}$ in $F'_0$ converges if and only if there exists a compact set $K \subset \mathbb{R}^n$ and $\varphi^{(k)}_m$ converges to $\varphi^{(k)}$ uniformly on $K$ for every $k \in \mathbb{N}_0$ as $m \to \infty$.

We already proved in Theorem 1.4 a relation between the support of a distribution and its S-asymptotics. Now, we shall complete this result.

We need a property of the S-asymptotics given in the next lemma.

**Lemma 1.1.** Let $\Gamma$ be a cone and let $\tilde{\Gamma}$ be a convex cone (it is partially ordered). A necessary and sufficient condition that for every $c(h) > 0$, $h \in \Gamma$

a) \(\lim_{\|h\| \to \infty, h \in \Gamma} T(x + h)/c(h) = 0 \quad \text{in} \quad F'_0,
\)

b) \(\lim_{h \to \infty, h \in \tilde{\Gamma}} T(x + h)/c(h) = 0 \quad \text{in} \quad F'_0,
\)

is that for every $\phi \in F'_0$ the following holds:

In case a): There exists $\beta(\phi) > 0$ such that

\[
\langle T(x + h), \phi(x) \rangle = 0, \quad \|h\| \geq \beta(\phi), \quad h \in \Gamma.
\] (1.15)

In case b): There exists $h_\phi \in \tilde{\Gamma}$ such that

\[
\langle T(x + h), \phi(x) \rangle = 0, \quad h \geq h_\phi, \quad h \in \tilde{\Gamma}.
\] (1.16)

**Proof.** We have to prove only that the condition is necessary in both cases. It is obvious that the condition is sufficient. Let us suppose the opposite, that is, the condition is not necessary. Then, we could find a sequence $\{a_m\}_m$ in $\Gamma$ such that in case a) $\|a_m\| \to \infty$ in $\Gamma$, and in case b) $a_m \to \infty$ in $\tilde{\Gamma}$, $(m \to \infty)$ and that

\[
\langle T(x + a_m), \phi(x) \rangle = a_m \neq 0, \quad m \in \mathbb{N}.
\]

Let find $c$ such that $c(h) = a_m$ for $h = a_m, m \in \mathbb{N}$. Clearly, for such a $c(h)$ the function $h \mapsto \langle T(x + h)/c(h), \phi(x) \rangle$ cannot converge to zero as $\|h\| \to \infty$, $h \in \Gamma$, or $h \to \infty$, $h \in \tilde{\Gamma}$. This contradicts our supposition that the S-asymptotics equals zero in both cases. $\square$
1. $S$-asymptotics in $F'_q$

Theorem 1.5. Let $\Gamma$ be a cone and let $T \in F'_0$. A necessary and sufficient condition that for every $r > 0$ there exists $\beta_r$ such that the sets

$$\text{supp } T \cap B(h, r), \ h \in \Gamma, ||h|| \geq \beta_r$$

are empty

is that $T(x + h) \sim c(h) \cdot 0$, $h \in \Gamma$ for every positive function $c$ on $\Gamma$.

Proof. Theorem 1.2 c) and Theorem 1.4 assert that the condition in Theorem 1.5 is necessary. We have to prove only that this condition is also sufficient.

Let us suppose that

$$\lim_{||h|| \to \infty, h \in \Gamma} T(x + h)/c(h) = 0 \ \text{in } F'_0.$$

Let $\phi \in F_0$. By Lemma 1.1 a), we know that there exists $\beta_0(\phi) = \inf \beta(\phi)$, where $\beta(\phi)$ is such that (1.15) holds. We shall prove that the set $\{\beta_0(\phi); \phi \in F_0, \text{supp } \phi \subset K\}$ is bounded for every compact set $K \subset \mathbb{R}^n$. Let us suppose the opposite. Then we could find a sequence $\langle \phi_k \rangle_k$ in $F_0, \text{supp } \phi_k \subset K$, $k \in \mathbb{N}$, and a sequence $\langle h_k \rangle_k$ in $\Gamma, ||h_k|| \to \infty$ such that

$$\langle T(t + h_k), \phi_k(t) \rangle = A_{k,p} = \begin{cases} a_k 
eq 0, & p = k \\ 0, & p < k. \end{cases}$$

We give the construction of sequences $\{\phi_k\}$ and $\{h_k\}$. Let $\phi_k \in F_0, \text{supp } \phi_k \subset K, k \in \mathbb{N}$, be such that $\langle \beta_0(\phi_k) \rangle_k$ is a strictly increasing sequence which tends to infinity. Then, there exist $\langle h_k \rangle_k$ in $\Gamma$ and $\varepsilon_k > 0, k \in \mathbb{N}$, such that $\beta_0(\phi_k - \varepsilon_k) \leq ||h_k|| \leq \beta_0(\phi_k) - \varepsilon_k$. Now, we shall construct the sequence $\langle \psi_p(t) \rangle_p$ in $F_0, \text{supp } \psi_p \subset K, p \in \mathbb{N}$, for which we have

$$\langle T(t + h_k), \psi_p(t) \rangle = \begin{cases} 0, & p \neq k \\ a_k, & p = k. \end{cases} \quad (*)$$

Put

$$\psi_p(t) = \phi_p(t) - \lambda^p_1 \phi_1(t) - \cdots - \lambda^p_{p-1} \phi_{p-1}(t), \ p > 1, t \in \mathbb{R}^n.$$

The sequence $\langle \lambda^p_i \rangle$, will be determined in such a way that $\psi_p(t)$ satisfies (*) the sought property, $p \in \mathbb{N}$.

It is easy to see that $\langle T(t + h_k), \psi_k(t) \rangle = a_k$ and $\langle T(t + h_k), \psi_p(t) \rangle = 0, k > p$. For a fixed $p$ and $k < p$ we can find $\lambda^p_i$, $i = 1, \ldots, p - 1$, such that for $k = 1, \ldots, p - 1$,

$$0 = \langle T(t + h_k), \psi_p(t) \rangle = A_{p,p} - \lambda^p_1 A_{k,1} - \cdots - \lambda^p_{p-1} A_{k,p-1}.$$
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Hence

$$\lambda^p_1 A_{k,1} + \cdots + \lambda^p_{p-1} A_{k,p-1} = A_{k,p}, \ k = 1, \ldots, p-1, \ p > 1.$$ 

Since $A_{k,k} \neq 0$ for every $k$, the system always has a solution.

We introduce a sequence of numbers $(b_k)_k$,

$$b_k = \sup \{ 2^k |\psi^{(i)}(t)|; \ i \leq k \}, \ k \in \mathbb{N}.$$ 

Then, the function

$$\psi(t) = \sum_{p=1}^{\infty} \frac{\psi_p(t)}{b_p}, \ t \in \mathbb{R}^n, \ \text{is in } \mathcal{F}_0 \ \text{and } \text{supp}\psi \subset K.$$ 

Thus, we have

$$\langle T(t+h_k), \psi(t) \rangle = \sum_{p=1}^{\infty} \langle T(t+h_k), \psi_p(t)/b_p \rangle = a_k/b_k.$$ 

Now, if we choose $c(h)$ such that $c(h_k) = a_k/b_k, \ k \in \mathbb{N}$, then

$$\frac{\langle T(t+h), \psi(t) \rangle}{c(h)}$$ does not converge to zero when $\|h\| \to \infty, h \in \Gamma$.

This proves that for every compact set $K$ there exists a $\beta_0(K)$ such that

$$\langle T(t+h), \phi(t) \rangle = 0, \ \|h\| \geq \beta_0(K), \ h \in \Gamma, \ \phi \in \mathcal{F}_0, \ \text{supp}\phi \subset K.$$ 

It follows that $T(t+h) = 0$ over $B(0,r), \|h\| \leq \beta(r), h \in \Gamma$ and $T(t) = 0$ over $B(h,r), \|h\| \geq \beta(r), h \in \Gamma$.  

Remark. The condition of Theorem 1.5 implies that the support of $T$ has the following property: The distance from supp $T$ to a point $h \in \Gamma, \ d(\text{supp} T, h)$, tends to infinity when $\|h\| \to \infty, h \in \Gamma$.

The next proposition shows that if in Definition 1.1 we take the limit (1.3) instead of the limit (1.2), then a more precise result is obtained.

**Theorem 1.6.** Let $T \in \mathcal{F}_0'$ and $\tilde{\Gamma}$ be an acute, open and convex cone (partially ordered, see Remark 1 after Definition 1.1). A necessary and sufficient condition for $\text{supp} T \in C_{\mathbb{R}^n}(a + \tilde{\Gamma})$ for some $a \in \tilde{\Gamma}$

$$\text{(1.17)}$$

is that

$$\lim_{h \in \Gamma, h \to \infty} T(x+h)/c(h) = 0 \ \text{in } \mathcal{F}_0' \ \text{for every } c(h) \ \text{(1.18)}$$

$$C_{\mathbb{R}^n} A = \mathbb{R}^n \setminus A.$$
1. S-asymptotics in $\mathcal{F}_g$

Proof. If (1.17) holds, then for any ball $B(0, 1)$, there exists a $h_r \in \hat{\Gamma}$ such that $B(h, r) \subset (a + \hat{\Gamma})$ for $h \geq h_r$, $h \in \hat{\Gamma}$. This implies (1.18), and by Lemma 1.1, (1.17) follows.

Let us suppose now that (1.16) and consequently (1.18) hold, but $f$ for any $a \in \tilde{\Gamma}$, $\text{supp}T \not\subset C_{\mathbb{R}^n}(a + \tilde{\Gamma})$. We fix such an $a = a_0 > 0$. There exists an $a_1 \in (a_0 + \tilde{\Gamma}) \cap \text{supp} T$. Since $\text{supp}T \not\subset C_{\mathbb{R}^n}(2a_1 + \tilde{\Gamma})$, there exists an $a_2 \in (2a_1 + \tilde{\Gamma} \cap \text{supp} T)$. In such a way, we construct a sequence $\langle a_k \rangle_k$ in $\tilde{\Gamma}$ such that $a_k \in (ka_k - 1 + \tilde{\Gamma}) \cap \text{supp} T$ and $a_k \geq ka_0$, $k \in \mathbb{N}$.

Since $a_k \in \text{supp}T, k \in \mathbb{N}$, it follows that there exists a sequence $\langle \phi_k \rangle_k$ in $\mathcal{F}_0$ such that $\text{supp} \phi_k \subset B(0, 1)$, $k \in \mathbb{N}$, and

$$\langle T(x + a_k), \phi_k(x) \rangle \neq 0, k \in \mathbb{N}.$$  

We put now:

$$c_{k,i} = \langle T(x + a_i), \phi_k(x) \rangle, \ k, i \in \mathbb{N};$$

$$b_k = \text{supp}\{2^k|\phi^{(j)}(x)|; j \leq k, x \in \mathbb{R}^n\}, k \in \mathbb{N}.$$  

We have to prove that there exists a sequence $\langle c_k \rangle_k$ such that $c_k \geq 1, k \in \mathbb{N}$ and

$$\sum_{k \geq 1} c_{k,i}/(b_k c_k) \neq 0, i \in \mathbb{N}. \quad (1.19)$$

First, we notice that $c_{k,k} \neq 0$. If $\sum_{k=1}^{\infty} c_{k,1}/b_k = 0$, then we take $c_1 > 1$, and if this series is different from 0, then we put $c_1 = 1$. Let $i = 2$. If $c_{1,2}/(b_1 c_1) + \sum_{k \geq 2} c_{k,2}/b_k = 0 (\neq 0)$, we take $c_2 > 1$ ($c_2 = 1$) such that

$$c_{1,1}/(c_1 b_1) + c_{2,1}/(c_2 b_2) + \sum_{k \geq 3} c_{k,1}/b_k \neq 0.$$  

Let $i = 3,$

$$c_{1,3}/(b_1 c_1) + c_{2,3}/(b_2 c_2) + \sum_{k \geq 3} c_{k,3}/b_k = 0 \text{ or } \neq 0.$$  

Then we take $c_3 > 1$ and $c_3 = 1$ respectively such that

$$c_{1,1}/(b_1 c_1) + c_{2,1}/(b_2 c_2) + c_{3,1}/(b_3 c_3) + \sum_{k=4}^{\infty} c_{k,1}/b_k \neq 0,$$
\[ c_{1,2}/(b_1 c_1) + c_{2,2}/(b_2 c_2) + c_{3,2}/(b_3 c_3) + \sum_{k=4}^{\infty} c_{k,2}/b_k \neq 0. \]

Continuing in this way, we construct a sequence \( \langle c_k \rangle \) for which (1.19) holds.

Let us put
\[ \psi_k(x) = \phi_k(x)/(b_k c_k), \quad k \in \mathbb{N} \]
and\[ \psi = \sum_{k \geq i} \psi_k. \]

From the properties of sequences \( (b_k) \) and \( (c_k) \), we can easily show that
\[ \sum_{k=1}^{N} \psi_k \to \psi \text{ in } \mathcal{F}_0 \text{ when } N \to \infty. \]

Relation (1.18) implies
\[ \langle T(x + a), \psi(x) \rangle = \sum_{k=1}^{\infty} c_{k,1}/(b_k c_k) \neq 0, \quad i \in \mathbb{N}. \]

We obtain that (1.18) does not hold for \( \psi \). This completes the proof. \( \square \)

In Theorem 1.6 the support of \( T \) can be just \( C_{R^n}(a + \tilde{\Gamma}) \). The question is: Is it possible to obtain a similar proposition for the S-asymptotics given by Definition 1.1? This question is analyzed in the next examples. We take \( \mathcal{F}_0 = \mathcal{D} \) in which the S-asymptotics by the weak and strong convergence are equivalent.

**Examples.** i) Let \( T(x, y) = \sum_{m \geq 1} m\delta(x - m, y) \). The given series converges in \( \mathcal{D}'(R^2) \). Since for a \( \phi \in \mathcal{D}(R^2) \), supp \( \phi \subset B(0, r) \), we have
\[ \lim_{n \to \infty} \left( \sum_{m=1}^{n} m\delta(x - m, y), \phi(x, y) \right) = \sum_{1 \leq m \leq r} m\phi(m, 0). \]

It follows that \( T \) is a distribution on \( R^2 \) (see Theorem XIII, Chapter 3 in [146]).

Let us remark that the support of \( T \) lies on the half line \( \gamma \equiv \{(p, 0) \in R^2; \; p > 0\} \). We can take for \( \Gamma \) the cone \( R^2_+ \equiv \{(\alpha, \beta) \in R^2; \; \alpha > 0, \beta > 0\} \). It is a convex, open and acute cone in \( R^2 \). We shall show that the limit
\[ \lim_{h \in \Gamma, ||h|| \to \infty} \langle T(u + h), \phi(u) \rangle \]
does not exist. To do this, it is enough to take the limit over the half line
\[ \gamma' \equiv \{(0, \alpha_0) + \gamma' \}, \alpha_0 > 0, \] which belongs to \( \Gamma \). If we choose \( \phi \) such that
\[ \phi > 0 \text{ and } \phi(0, -\alpha_0) = 1, \]
then for \( h = (p, \alpha_0) \in \gamma' \)
\[ \langle T(u + h), \phi(u) \rangle = \sum_{m \geq 1} m \phi(m - p, -\alpha_0) \geq p. \]
Note, if \( \|h\| \to \infty \), then \( p \to \infty \), as well. Consequently, the answer to the
posed question is negative.

ii) The following example shows that if
\[ \lim_{h \in \tilde{\gamma}, \|h\| \to \infty} T(x + h)/c(h) = 0 \text{ in } D' \]
for every positive \( c(h) \), and every \( \tilde{\gamma} = \{pw, p > 0\}, w \in \Gamma \), then this does
not imply that
\[ \lim_{h \in \Gamma, \|h\| \to \infty} T(x + h)/c(h) = 0 \text{ in } D'. \]
Let us remark that both limits on a \( \tilde{\gamma} \), when \( h \to \infty \) or \( \|h\| \to \infty \), are equal.

Let \( T \) be given by
\[ T(x, y) = \sum_{m \geq 1} m \delta(x - m, y - \frac{1}{m}) \text{ on } \mathbb{R}^2. \]
The support of \( T \) lies on the curve \( \{(x, 1/x); x > 0\} \). Let \( \Gamma = \mathbb{R}^2_+ \), \( w = (\cos \alpha, \sin \alpha), 0 < \alpha < \frac{\pi}{2} \) and \( \tilde{\gamma} = \{pw; p > 0\} \). Then, for a \( \phi \in \mathcal{D}(\mathbb{R}^2) \)
\[ (T(x+h_1, y+h_2), \phi(x, y)) = \sum_{m \geq 1} m \phi \left( m - h_1, \frac{1}{m} - h_2 \right) \to 0, \|h\| \to \infty, h \in \tilde{\gamma}. \]
In order to show that
\[ \lim_{h \in \Gamma, \|h\| \to \infty} \langle T(x + h_1, y + h_2), \phi(x, y) \rangle \]
does not exist, take \( h \) to belong to \( \gamma = \{(x, a); x > 0\} \) for a fixed \( a > 0 \), as
we did in example i).

**Remarks.** As a consequence of Theorem 1.5 and Theorem 1.6, we have
some results concerning the convolution product (see [135], p. 102).

Let
\[ G_1 \equiv \{f \in C^\infty; \text{ supp } f \subset C_{\mathbb{R}^n}; \|h\| \geq h_f \}, \]
\[ G_2 \equiv \{f \in C^\infty; \text{ supp } f \subset C_{\mathbb{R}^n}; \|h\| \geq \tilde{h_f} \}. \]

**Corollary 1.1.** For a fixed \( T \in D' \), the convolution \( T * \phi \) maps \( D \) into \( G_1 \)
if and only if the support of \( T \) has the property given in Theorem 1.5.

**Corollary 1.2.** For a fixed \( T \in D' \) and for a convex, open and partially
ordered cone \( \tilde{\Gamma} \) the convolution \( T * \phi \) maps \( D \) into \( G_2 \) if and only if \( \text{ supp } T \subset C_{\mathbb{R}^n}(a + \tilde{\Gamma}) \) for some \( a \in \tilde{\Gamma} \).
1.8 Characterization of some generalized function spaces

**Theorem 1.7.** A necessary and sufficient condition for a distribution \( T \) to belong to:

a) \( \mathcal{E}' \) is that \( T(x+h) \sim c(h) \cdot 0, h \in \mathbb{R}^n \) for every positive \( c \).

b) \( \mathcal{O}'_c \) is that \( T \) has S-asymptotic behavior related to every \( c(h) = \|h\|^{-\alpha}, \alpha \in \mathbb{R}_+ \) and with limit \( U_c = 0 \).

c) \( \mathcal{B}' \) is that \( T \) has S-asymptotic behavior related to every positive \( c, c(h) \to \infty, \|h\| \to \infty \), and with limit \( U_c = 0 \).

**Proof.**
a) It is a direct consequence of Theorem 1.5.

b) It is enough to apply Theorem IX Chapter VII in [146] which states: A necessary and sufficient condition that a distribution \( T \) belongs to \( \mathcal{O}'_c \) is that for every \( \varphi \in \mathcal{D} \) the function \( (T \ast \varphi)(h) \) is continuous and of fast descent at infinity. (see 0.5.1).

c) By Theorem XXV, Chapter VI in [146] a distribution \( T \in \mathcal{B}' \) if and only if \( T \ast \varphi \in L^\infty(\mathbb{R}^n) \) for every \( \varphi \in \mathcal{D} \). Suppose that \( T \in \mathcal{B}' \). Then for every \( \varphi \in \mathcal{D} \) and \( c(h) \to \infty, \|h\| \to \infty \)

\[
\lim_{h \in \mathbb{R}^n, \|h\| \to \infty} \left( \frac{T(x+h)}{c(h)} , \varphi(x) \right) = \lim_{h \in \mathbb{R}^n, \|h\| \to \infty} \left( \frac{T \ast \varphi(h)}{c(h)} \right) = 0.
\]

Suppose that \( (T \ast \varphi)(h)/c(h) \to 0, \|h\| \to \infty \), for every \( \varphi \in \mathcal{D} \) and for every \( c, c(h) \to \infty \) as \( \|h\| \to \infty \). We will show that \( (T \ast \varphi)(h) \in L^\infty(\mathbb{R}^n) \) for every \( \varphi \in \mathcal{D} \). Then, by the same theorem, it follows that \( T \in \mathcal{B}' \).

Let us assume the contrary, i.e., that \( (T \ast \varphi_0)(h) \) is not bounded for a \( \varphi_0 \in \mathcal{D} \). Then, we could find two sequences \( \langle h_m \rangle \) in \( \mathbb{R}^n \) and \( \langle c_m \rangle \) in \( \mathbb{R} \) such that \( |c_m| \to \infty \) as \( m \to \infty \), \( \|h_m \| \geq m \) and \( (T \ast \varphi_0)(h_m) = c_m \).

Now, for \( c_0(h) \) such that \( c_0(h_m) = \sqrt{|c_m|}, m \in \mathbb{N} \), the limit \( <T(x+h)/c_0(h), \varphi_0(x)> \) would not exist, as \( \|h\| \to \infty \). This is in a contradiction with our assumption that \( T \) has the S-asymptotics related to every \( c(h) \) which tends to infinity as \( \|h\| \to \infty \). □

**Proposition 1.7.** a) If for every rapidly decreasing function \( c \), \( T \) has the S-asymptotic behavior related to \( c^{-1} \) and with limit \( U_c \) (\( U_c = 0 \) is included), then \( T \in \mathcal{S}' \).
1. \textit{S-asymptotics in $F'_g$}

b) If for every rapidly exponentially decreasing function $c$ (for every $k > 0$, $c(h) \exp(k\|h\|) \to 0$, $\|h\| \to \infty$) a distribution $T$ has the S-asymptotic behavior related to $c^{-1}$ with limit $U_c$ ($U_c = 0$ is included), then $T \in K'_1$.

Proof. a) Let $c$ be given. There exists $\beta_0$ such that for every $h$, $\|h\| \geq \beta_0$, and for every $\varphi \in D$

$$|< T(x + h) \cdot c(h), \varphi(x) >| \leq |< U_c, \varphi >| + \varepsilon_\varphi \leq M_\varphi + \varepsilon_\varphi.$$  

Therefore, the set $\{T(x + h) \cdot c(h); h \geq \beta_0\}$ is weakly bounded and thus bounded in $D'$. By Theorem VI 4°, Chapter VII in [146] if $\{c(h)T(x + h); \|h\| > \beta_0\}$ is bounded in $D'$, for every $c$ of fast descent, then $T \in S'$.

b) The proof is similar to that of a), if we use the following theorem proved which will be showed below (cf. Theorem 1.15):

Let $T \in D'$. If for every rapidly exponentially decreasing function $r$ on $\mathbb{R}^n$ the set $\{r(h)T(x + h); h \in \mathbb{R}^n\}$ is bounded in $D'$, then $T \in K'_1$. $\square$

1.9 \textit{Structural theorems for S-asymptotics in $F'$}

In the analysis of the S-asymptotics and its applications, it is useful to know analytical expression for generalized functions having S-asymptotics, especially if it is given via continuous functions and their derivatives. The next theorems are of this kind. These types of theorems are usually referred as structural theorems.

Our first result concerns the one-dimensional case, a refinement will be obtained in Theorem 1.9 below.

\textbf{Theorem 1.8.} Let $\alpha \in \mathbb{R}$ and let $L$ be a slowly varying function. Suppose that $f \in D'(\mathbb{R})$. If $f(x + h) \sim c(h) \cdot e^{\alpha x}, h \in \mathbb{R}_+$, then there is an $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$ the following holds:

1. Let $\alpha \neq 0$ and $c(h) = e^{\alpha h}L(e^h)$, $h > 0$. Then there are $g_{m,i} \in C(1, \infty), i = 0, 1, \ldots, m$ such that

$$f(x) = \sum_{i=0}^{m} g_{m,i}(x), \quad x \in (1, \infty),$$

and

$$g_{m,i}(x) \sim C_i x^m \exp(\alpha x)L(\exp x), \quad x \to \infty, \quad i = 0, \ldots, m,$$

where $C_i$ are suitable constants.
(2) Let \( \alpha = 0 \) and \( c(h) = h^\nu L(h), \, h > 0 \). Then,

- a) If \( \nu > -1 \), then there is \( F_m \in C(1, \infty) \) such that \( f = F_m^{(m)} \) and \( F_m(x) \sim x^{m+\nu}L(x), \, x \to \infty; \)

- b) If \( \nu \leq -1 \), then there are \( f_{m,i} \in C(1, \infty) \) and \( A_{m,i} \neq 0, \, i = 0, 1, \ldots, m \), such that
  
  \[
  f_{m,i}(x) \sim A_{m,i}x^{m+\nu-i}L(x), \, i = 0, 1, \ldots, m
  \]

and

\[
 f(x) = \sum_{i=0}^{m} f_{m,i}^{(m-i)}(x), \, \, \, x \in (1, \infty).
\]

Proof. Case \( \alpha \neq 0 \). By the remark 1. after Proposition 1.2, there exists a function \( \tilde{c} \in C^\infty(\mathbb{R}) \) such that \( c(h)/\tilde{c}(x+h) \to \exp(-\alpha x), \, h \to \infty \) in \( \mathbb{R} \) and \( \tilde{c}^{(i)}(x+h)/c(h) \to \alpha^i \exp(\alpha x), \, i \in \mathbb{N}_0. \)

Since \( f(x+h)/c(h) \to \exp(\alpha x) \) with strong convergence and the set \( \{ c(h)\phi(x)/\tilde{c}(x+h); \, h \geq A \} \) is bounded in \( \mathcal{D}(\mathbb{R}) \), we can then use Theorem XI, Chapter III in [146] to obtain

\[
 \lim_{h \to \infty} \left( \frac{f(x+h)}{\tilde{c}(x+h)} - \phi(x) \right) = \lim_{h \to \infty} \left( \frac{f(x+h)}{c(h)} \cdot \frac{c(h)}{\tilde{c}(c+h)} \phi(x) \right) = (1, \phi(x)), \, \phi \in \mathcal{D}(\mathbb{R}).
\]

Let \( \theta \in C^\infty, \theta(x) = 0 \) for \( x < 0 \) and \( \theta(x) = 1 \) for \( x > 1 \). We have

\[
 \frac{\theta(x+h)f(x+h)}{\tilde{c}(x+h)} \to 1 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}) \quad \text{as} \quad h \to \infty.
\]

Thus, \( \{ \theta(x+h)f(x+h)/\tilde{c}(x+h); \, h > 0 \} \) is a bounded subset of \( \mathcal{D}'(\mathbb{R}) \). This implies that this set is bounded in \( \mathcal{S}'(\mathbb{R}) \) as well (cf. Theorem XXV, Chapter VI in [146]). Since \( \mathcal{D}(\mathbb{R}) \) is dense in \( \mathcal{S}(\mathbb{R}) \), by the Banach–Steinhaus theorem, we obtain

\[
 \frac{\theta(x+h)f(x+h)}{\tilde{c}(x+h)} \to 1 \quad \text{in} \quad \mathcal{S}'(\mathbb{R}) \quad \text{as} \quad h \to \infty, \quad \text{i.e.,}
\]

\[
 \lim_{h \to \infty} \left\langle \frac{\theta f/\tilde{c}(x+h)}{d(h)}, \phi(x) \right\rangle = (1, \phi), \quad \text{for every} \quad \phi \in \mathcal{S}(\mathbb{R}),
\]

where \( d(h) = 1, \, h > A. \) We shall now borrow two theorems about quasi-asymptotics (see Definition 2.2) which will be shown later. Since the S-asymptotics in \( \mathcal{S}'_+ \) with \( \nu > -1 \) implies the quasi-asymptotics of \( \theta f/\tilde{c} \).
(see Theorem 2.47), the structural theorem for the quasi-asymptotics (see Theorem 2.2) implies that there is \( m_0 \in \mathbb{N} \) such that for every \( m > m_0 \) there is \( F_m \in C(\mathbb{R}) \) such that
\[
(\theta f / \tilde{c})(x) = F_m^{(m)}(x), \quad x \in \mathbb{R},
\]
and \( F_m(x) \sim x^m \) as \( x \to \infty \). Thus, we obtain
\[
f(x) = \tilde{c}(x) F_m^{(m)}(x), \quad x \in (1, \infty).
\]
The Leibnitz formula implies
\[
f(x) = \sum_{i=0}^{m} \binom{m}{i} (-1)^i (\tilde{c}^{(i)}(x) F_m(x))^{(m-i)}, \quad x \in (1, \infty).
\]
Since
\[
\frac{\tilde{c}^{(i)}(x+h)}{c(h)} \to \alpha^i e^{\alpha x}, \quad h \to \infty, \quad x \in \mathbb{R},
\]
we obtain
\[
\tilde{c}^{(i)}(h) \sim \alpha^i c(h), \quad h \to \infty.
\]
This implies the result when \( \alpha \neq 0 \) and \( c(h) = e^{\alpha h} L(h), h > 0 \).

In case \( \alpha = 0 \) and \( c(h) = h^\nu L(h), \quad h > 0, \nu > -1 \), the \( S \)-asymptotics of \( \theta f \) related to \( c(h) = h^\nu L(h), h > 0, \nu > -1 \), implies the quasi-asymptotics of \( \theta f \) related to the same \( c(h) \) (see Theorem 2.47). The assertion follows now from Theorem 2.2.

If \( \nu \leq -1 \), then we take \( k > 0 \) such that \( k + \nu > -1 \). With \( \theta \) as in the preceding proof and by Theorem 1.2 b), we have
\[
(1 + (x+h)^2)^{k/2} \theta(x+h) f(x+h) \sim h^{k+\nu} L(h) \cdot 1, \quad h \in \mathbb{R}_+.
\]
By the same arguments as in the preceding proof, we have that there is \( m_0 \in \mathbb{N} \) such that for every \( m > m_0 \) there is an \( F_m \in C(\mathbb{R}), \supp F_m \subset (0, \infty) \)
\[
F_m(x) \sim x^{\nu + k + m} L(x), \quad x \to \infty
\]
\[
(1 + x^2)^{k/2} \theta(x) F_m = F_m^{(m)}(x), \quad x \in \mathbb{R}.
\]
Thus, for \( x \in (1, \infty) \),
\[
f(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \left( \frac{1}{(1 + x^2)^{k/2}} \right)^{(i)} F_m^{(m-i)}(x).
\]
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The result now follows from the fact
\[
\left( \frac{1}{(1 + x^2)^{k/2}} \right)^{(i)} \sim C_i x^{-k-i}, \quad x \to \infty,
\]
where \( C_i \neq 0 \) are suitable constants, \( i = 0, \ldots, m \).

We can show a more general result which describes the precise structure of a distribution having S-asymptotics.

**Theorem 1.9.** If \( T \in D' \) has S-asymptotics related to the open cone \( \Gamma \) and the continuous and positive function \( c(h), h \in \Gamma \), then for the ball \( B(0,r) \) there exist continuous functions \( F_i, |i| \leq m \), such that \( F_i(x + h)/c(h) \) converges uniformly for \( x \in B(0,r) \) when \( h \in \Gamma, \|h\| \to \infty \), and the restriction of the distribution \( T \) on \( B(0,r) + \Gamma \) can be given in the form \( T = \sum_{|i| \leq m} D^i F_i \).

We need the following lemma for the proof of the theorem:

**Lemma 1.2.** Let \( T \in D' \), \( T(x+h) \sim c(h) \cdot U(x), h \in \Gamma \). Then, for an open ball \( B(0,r) \) and a relatively compact open neighborhood \( \Omega \) of zero in \( \mathbb{R}^n \), there exists an \( m \in \mathbb{N}_0 \) such that for every \( \varphi, \psi \in D^n_0 \) the function \( (T \ast \varphi \ast \psi)(x) \) is continuous for \( x \in B(0,r) + \Gamma \). Moreover, the set of functions \( \{\{T_h \ast \varphi \ast \psi)(x); h \in \Gamma\}\} \) converges uniformly for \( x \in B(0,r) \) to \((U \ast \varphi \ast \psi)(x)\), as \( h \in \Gamma, \|h\| \to \infty; T_h = T(x+h)/c(h) \).

**Proof.** Suppose that \( T \) has S-asymptotics related to \( c(h) \). Then, the set \( \{T(x+h)/c(h) \equiv T_h; h \in \Gamma\} \) is weakly bounded in \( D' \) and consequently, bounded in \( D' \). A necessary and sufficient condition that a set \( B' \subset D' \) is bounded in \( D' \) is: for every \( \alpha \in \mathcal{D} \) the set of functions \( \{T \ast \alpha; T \in B'\} \) is bounded on every compact set \( K \subset \mathbb{R}^n \) (see §7, Chapter VI in [146]). Moreover \( \{T \ast \alpha; T \in B'\} \) defines a bounded set of regular distributions.

Let \( C(\Omega) = K \) (\( C(\Omega) \) is the closure of \( \Omega \)); \( K \) is a compact set. For a fixed \( \alpha \in C^n_0 \), supp \( \alpha \subset K \), the linear mappings \( \beta \mapsto (T_h \ast \alpha) \ast \beta, h \in \Gamma \), are continuous mappings of \( D_K \) into \( \mathcal{E} \) because of the separate continuity of the convolution. Since the set \( \{T_h \ast \alpha; h \in \Gamma\} \) is a bounded set in \( D' \), for every ball \( B(0,r) \) the set of mappings \( \beta \to \{(T_h \ast \alpha) \ast \beta; h \in \Gamma\} \) is the set of equicontinuous mappings of \( D_K \) into \( L^n_B \), where \( B = B(0,r) \). Now there exists an \( m \geq N_0 \) such that the linear mappings \( (\alpha, \beta) \to T_h \ast \alpha \ast \beta \) which map \( D_K \times \mathcal{D}_K \) into \( L^n_B \) can be extended to \( D^{m,n}_K \times \mathcal{D}^n_0 \) in such a way that \( (\alpha, \beta) \to T_h \ast \alpha \ast \beta, h \in \Gamma \), are equicontinuous mappings of \( D^{m,n}_K \times \mathcal{D}^n_0 \) into \( L^n_B \) (see for example the proof of Theorem XXII, Chapter VI in [146]).
1. S-asymptotics in $F'_q$

We proved that for every $\varphi, \psi \in D^m$ and every $h \in \Gamma$ the functions $x \mapsto (T_h \ast \varphi \ast \psi)(x)$ are continuous in $x \in B(0, r)$. From the relation $(T_h \ast \varphi \ast \psi)(x) = (T \ast \varphi \ast \psi)(x + h)/c(h)$ and from the properties of $c$ it follows that $y \mapsto (T \ast \varphi \ast \psi)(y)$ is a continuous function for $y \in B(0, r) + \Gamma$ and $\varphi, \psi \in D^m$. 

It remains to prove that $T_h \ast \varphi \ast \psi$ converges to $U \ast \varphi \ast \psi$ as $\|h\| \to \infty$, $h \in \Gamma$, in $L^2$. We know that $D$ is a dense subset of $D^m$, $m \geq 0$. We can construct a subset $A$ of $D_K$ to be dense in $D^m$. The set of functions $T_h \ast \alpha \ast \beta$ converges in $L^2$ for $\alpha, \beta \in A$, when $h \in \Gamma$, $\|h\| \to \infty$. Taking care of the equicontinuity of the mappings $D^m \times D^m \to L^\infty$, defined by $T_h \ast \varphi \ast \psi$, we can use the Banach–Steinhaus theorem to prove that $T_h \ast \varphi \ast \psi$ converges in $L^2$ when $h \in \Gamma$, $\|h\| \to \infty$.

Proof of the Theorem. We shall use (VI, 6; 23) from [146].

$$\Delta^k \ast (\gamma E \ast \gamma E \ast T) - 2\Delta^k \ast (\gamma E \ast \xi \ast T) + (\xi \ast \xi \ast T) = T,$$  \hspace{1cm} (1.20)

where $E$ is a solution of the iterated Laplace equation $\Delta^k E = \delta; \gamma, \xi \in D_\Omega$. We have only to choose the natural number $k$ large enough so that $E$ belongs to $D^m$. Now, it is possible to take $F_1 = \gamma E \ast \gamma E \ast T, F_2 = \gamma E \ast \xi \ast T$ and $F_3 = \xi \ast \xi \ast T$. All of these functions are of the form $F_i = T \ast \varphi_i \ast \psi, \varphi_i, \psi, \in D^m, i = 1, 2, 3$.

The following holds:

$$F_i(x + h)/c(h) = (F_i(x)/c(h)) \ast \tau_{-h} = (T \ast \varphi_i \ast \psi_i)(x) \ast \tau_{-h}/c(h)$$

$$= (T(x) \ast \tau_{-h})/c(h) \ast (\varphi_i \ast \psi_i) = (T_h \ast \varphi_i \ast \psi_i)(x), \; x \in B(0, r), \; h \in \Gamma.$$  

Hence, by Lemma 1.2 it follows that $F_i(x + h)/c(h)$ converges uniformly for $x \in B(0, r)$ when $h \in \Gamma$, $\|h\| \to \infty$.  

Consequences of Theorem 1.9

a) If the functions $F_i$, $|i| \leq m$, have the property given in Theorem 1.9 and if $\Gamma = R^n$, then the regular distributions defined by the functions $F_i/c$ have the S-asymptotic behavior related to $c_i(h) \equiv 1$.

b) If $T \in S'$, then all functions $F_i$, $|i| \leq m$, are continuous for $x \in B(0, r + \Gamma)$, and of slow growth ($F_i = (1 + r^2)^q f_i, |i| \leq m, q \in R$, where $r = \|x\|$ and $f_i, |i| \leq m$, are continuous and bounded functions).

c) The converse of Theorem 1.9 is also true. Therefore, it completely characterizes those distributions having S-asymptotics.
Asymptotic Behavior of Generalized Functions

Proof. a) Suppose that $\Gamma = \mathbb{R}^n$. Then by the properties of $F_i, |i| \leq m$, the functions $F_i/c$ are continuous and $F_i(x)/c(x)$ converge to the numbers $C_i$ as $\|x\| \to \infty$. Now, for $|i| \leq m$ we have

$$\lim_{\|h\| \to \infty} (F_i(x+h)/c(x+h), \varphi(x)) = \lim_{\|h\| \to \infty} \int_{\mathbb{R}^n} (F_i(x+h)/c(x+h))\varphi(x)dx = (C_i, \varphi), \varphi \in \mathcal{D}.$$

b) If $T \in \mathcal{S}'$, then there exists a $q \in \mathbb{R}$ such that the set of distributions $\{T(x+h)/(1 + \|h\|^q); h \in \mathbb{R}^n\} = W$ is bounded in $\mathcal{D}'$ (Theorem VI, Chapter VII in [146]). We can now repeat the first part of the proof of Lemma 1.2 but with $c(h) = (1 + \|h\|^q)$, $h \in \Gamma = \mathbb{R}^n$. In this way we may obtain that there exists a $p \in \mathbb{N}_0$ such that for $\varphi, \psi \in \mathcal{D}^p_\Omega$ and $x \in \mathbb{R}^n$ the function $x \mapsto (T*\varphi*\psi)(x)$ is continuous and $(T*\varphi*\psi)(x)/(1 + \|x\|^q), x \in \mathbb{R}$ is bounded. It remains only to choose the number $k$ in (1.20) large enough so that $\gamma E \in \mathcal{D}^k_{\Omega}$. 

c) It follows directly from Theorem 1.2 a) and Proposition 1.5. \qed

We can also characterize the structure of ultradistributions having $S$-asymptotics.

**Theorem 1.10.** ([132]). Let $T \in \mathcal{D}^*_\Omega$. Suppose: (M.1), (M.2) and (M.3) are satisfied by $M_p$; $\Omega = \Omega_1 + \Gamma_1$, where $\Omega_1 \subset \mathbb{R}^n$ is open set and $\Gamma_1 \subset \mathbb{R}^n$ is convex cone; $\Gamma$ is a subcone of $\Gamma_1$. Then $T$ has the $S$-asymptotics related to $c$ and $\Gamma$ if and only if for a given open and relatively compact set $A(\overline{A} \subset \Omega)$, there exist an ultradifferential operator $P(D)$ of class $*$ and continuous functions $f_1$ and $f_2$ on $A + \Gamma$ such that

$$\lim_{h \in \Gamma, \|h\| \to \infty} f_i(x+h)/c(h), \; i = 1, 2,$$

exist uniformly in $x \in A$, $i = 1, 2$, and $T = P(D)f_1 + f_2$ on $A + \Gamma$.

**Proof.** One can easily prove that the condition is sufficient.

The condition is necessary. Suppose that $T(x+h) \sim c(h) \cdot U(x), \; h \in \Gamma$ in $\mathcal{D}^*_\Omega$. Denote by $T_h = \tau_h T/c(h)$ and by $(CB)_A$ the space of continuous and bounded functions on $A$. By Theorem 6.10 in [79], $F_h : \varphi \mapsto T_h \ast \varphi, \; h \in \Gamma$, are continuous mappings: $\mathcal{D}^*_h \to (CB)_A$. Consequently, $F_h$ are the continuous mappings: $\mathcal{D}^*_h \to (CB)_A$, where $K = \overline{E} \subset \Omega$. The set $\{T_h; h \in \Gamma, \|h\| \geq \gamma\}, \gamma > 0$, is bounded in $\mathcal{D}^*_\Omega$ because of the $S$-asymptotics of $T$. 

Thus, for a fixed $\varphi \in \mathcal{D}_K'$, the set $\{ F_h(\varphi) \colon h \in \Gamma, \|h\| \geq \gamma \}$ is bounded in $(CB)_A$. Since $\mathcal{D}_K'$ is a barrelled space, by the Banach theorem it follows that the family of functions $\{ F_h \} = \{ F_h ; h \in \Gamma, \|h\| \geq \gamma \}$ is equicontinuous. Therefore, there exists $H_p$, $p \in \mathbb{N}$ of the form (0.1) and $\beta > 0$ such that any function in $\{ F_h \}$ maps the neighborhood of zero

$$V_\beta = \left\{ \phi \in \mathcal{D}_K^* ; \sup_{x \in K, \|a\| \in \mathbb{N}_0} |\phi(a)(x)|/H_{\|a\|} M_{\|a\|} < \beta \right\} \quad (1.21)$$

into the unit ball $B(0, 1)$ in $(CB)_A$. Denote by $\tilde{D}_K^H_{\|M\|}$ the completion of $\mathcal{D}_K'$ under the norm $q_{H, M}$ (see (0.5)).

Using the extension of a function through its continuity (see [12]), we shall show that the family $\{ F_h \}$ can be extended on $\tilde{D}_K^H_{\|M\|}$, keeping the uniform continuity; let us denote it by $\{ \overline{F}_h \}$.

Let $\phi \in \tilde{D}_K^H_{\|M\|}$ and let $(\phi_j)_j$ be a sequence in $\mathcal{D}_K'$ which converges to $\phi$ being a Cauchy sequence in the norm $q_{H, M}$. We shall prove that $T_h * \psi_j$ converges, as $j \to \infty$, in $(CB)_A$ uniformly in $h \in \Gamma$, $\|h\| \geq \gamma$. It is enough to prove that $(T_h * \psi_j)_j$ is a Cauchy sequence in $(CB)_A$.

Every neighborhood of zero $W$ in $(CB)_A$ contains the ball $B(0, \delta)$ for some $\delta > 0$. The neighborhood of zero $V_{\beta \delta}$, given by (1.21), satisfies $T_h * V_{\beta \delta} \subset W$, when $h \in \Gamma$, $\|h\| \geq \gamma$.

Let $j_0 \in \mathbb{N}_0$ be such that $\psi_i - \psi_j \in V_{\beta \delta}$ if $i, j \geq j_0$. Then,

$$T_h * \psi_i - T_h * \psi_j = T_h * (\psi_i - \psi_j) \in W, \; h \in \Gamma, \; \|h\| \geq \gamma.$$  

This proves the existence of the family $\{ \overline{F}_h \}$ and that $\lim_{j \to \infty} T_h * \psi_j = g_h \in (CB)_A$, $\|h\| \geq \gamma$. We shall prove that $g_h = T_h * \phi$. The sequence $\{ \psi_j \}_j$ converges to $\phi$ also in $\mathcal{D}_K^*$, as well. So, $T_h * \psi_j$ converges to $T_h * \phi$ as $j \to \infty$, in $\mathcal{D}_K^*$, where $\Omega = \omega - B_{r+\varepsilon}$ (see [79], p. 73). Thus, $(T_h * \phi)_A$ must be $g_h$.

It remains to prove that for $\phi \in \tilde{D}_K^H_{\|M\|}$, $T_h * \phi$ converges in $(CB)_A$ as $h \in \Gamma, \|h\| \to \infty$. Since $\{ \overline{F}_h \}$ is an equicontinuous family of functions, for every $\phi \in \tilde{D}_K^H_{\|M\|}$ the set $\{ T_h * \phi ; h \in \Gamma, \|h\| \geq \gamma \}$ is bounded in $(CB)_A$. By the Banach–Steinhaus theorem $T_h * \phi$ converges in $(CB)_A$.

Now we will give the analytic form of $T$. Let $\phi \in \mathcal{D}_Q$ such that $Q$ is a compact subset of the interior of $K$ and $q_{H, M}(\phi) < \infty$. One can easily prove that $\phi \in \tilde{D}_K^H_{\|M\|}$. Now, by Theorem 2.11 in [81], $T = P(D)(\phi * T) - w * T$, where we can take that $\phi \in \tilde{D}_K^H_{\|M\|}$ and $w \in \mathcal{D}_K'$. Let us denote $\phi * T$
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by \( f_1 \) and \( w * T \) by \( f_2 \). The properties of these two functions follow by the previous part of the proof. \(\square\)

We pass now to the case of Fourier hyperfunctions.

**Theorem 1.11.** Let \( f = [F'] \in Q(D^n) \), \( F' \in \hat{O}((D^n + iI^#D^n)) \). If \( f \) has the \( S \)-asymptotics related to \( c \) and \( \Gamma \), then there exist an elliptic local operator \( J(D) \) and functions \( q_s \in C^\infty(R^n) \), \( s \in \Lambda \), of infra exponential type such that:

1. \( q_s(z) \in \hat{O}(D^n + iI^s) \), where \( D^n + iI^s \) is an infinitesimal wedge of type \( D^n + i\Gamma_{s0} \), \( s \in \Lambda \).

2. \( f = J(D) \sum_{s \in \Lambda} q_s(x) \).

3. For every \( s \in \Lambda \) there exists \( \varepsilon_0 > 0 \) such that \( q_s(x + i\varepsilon s)/c(x) \), \( s \in \Lambda \) converge as \( \|x\| \to \infty \), \( x \in \Gamma \), for every fixed \( \varepsilon \), \( 0 < \varepsilon < \varepsilon_0 \).

**Proof.** Let

\[
\int [f] \cong \sum_{\sigma \in \Lambda} F_{\sigma}(x + i\Gamma_{\sigma}), \quad F_{\sigma} = \text{sgn} F_{\sigma}'
\]

and

\[
F(f) = [R] \cong \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} \mathcal{F}(\chi_{\sigma} F_{\sigma})(\xi - i\Gamma_{\tilde{\sigma}} 0).
\]

Then there exists a monotone increasing continuous positive valued function \( \varepsilon(r) \), \( r \geq 0 \), which satisfies \( \varepsilon(0) = 1 \), \( \varepsilon(r) \to \infty \), \( r \to \infty \) and such that

\[
|R(\zeta)| \leq C_k \exp(|\zeta|/\varepsilon(|\zeta|), \quad \frac{1}{k} \leq |\text{Im} \, \zeta_j| \leq 1,
\]

where \( j = 1, \ldots, n \) and \( k \in N \) (cf. [74], p. 652).

By Lemma 1.2 in [73], we can choose an entire function \( J \) of infra exponential type on \( C^n \) which satisfies the estimate:

\[
|J(\zeta)| \geq C \exp(|\zeta|/\varepsilon(|\zeta|)), \quad |\text{Im} \, \zeta| \leq 1.
\]

Then \( J^{-2}(\zeta) \in \hat{O}(D^n + i\{|\mu|<1\}) \) and

\[
|R(\zeta)/J^2(\zeta)| \leq C_k \exp(-|\zeta|/\varepsilon(|\zeta|)) \leq C_k \exp(-|\zeta|^{\gamma}),
\]

where \( \frac{1}{k} \leq |\text{Im} \, \zeta_j| \leq 1, \ j = 1, \ldots, n, \) and \( 0 < \gamma < 1 \).
1. S-asymptotics in $\mathcal{F}_g$

Denote by $g = F^{-1}(1/J^2)$. By Theorem 8.2.6 in [75], $g \in Q^{-1}(D^n)$. By Proposition 8.4.3. in [75]

$$f \ast g = \int_{\mathbb{R}} f(\cdot - t)g(t)dt \in Q(D^n)$$

and

$$\mathcal{F}(f \ast g) = \mathcal{F}(f)\mathcal{F}(g).$$

By the Corollaries of Proposition 2 in [165]

$$I = (-\alpha, \alpha)^n.$$  

We can always assume that there exists $\alpha \in \mathbb{R}_+$ such that

$$I = (-\alpha, \alpha)^n.$$  

Then we denote by $I_\sigma = I \cap \Gamma_\sigma$, $\sigma \in \Lambda$.

By Proposition 8.3.2 in [75] the following assertions are true:

$F_\sigma \chi_\tilde{\sigma} \in O(x + iI_\sigma)$ and it decreases exponentially outside any cone containing $\Gamma_\tilde{\sigma}$ as a proper subcone.

$F(F_\sigma \chi_\tilde{\sigma}) \in O(x - iI_\sigma)$ and it decreases exponentially outside any cone containing $\Gamma_\tilde{\sigma}$ as a proper subcone.

$F(F_\sigma \chi_\tilde{\sigma})J^{-2}$ has the same cited properties as $F(F_\sigma \chi_\tilde{\sigma})$.

$F(F_\sigma \chi_\tilde{\sigma})J^{-2} \chi_\sigma \in O(x - iI_\sigma)$ and it decreases exponentially outside any cone containing $\Gamma_\sigma$ and $\Gamma_\tilde{\sigma}$ proper subcone.

$F^{-1}(F(F_\sigma \chi_\tilde{\sigma})J^{-2} \chi_\sigma) \in O(x + i(I_\sigma \cup I_\tilde{\sigma}))$ and it decreases exponentially outside any cone containing $\Gamma_\sigma$ as a proper subcone.

Consider now the Fourier hyperfunction $f \ast g$ given in (1.22):

$$f \ast g = F^{-1}(\mathcal{F}(f)\mathcal{F}(g))$$

$$= \frac{1}{(2\pi)^n} \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} \int_{\mathbb{R}^n} e^{iz_\sigma \zeta} \frac{\mathcal{F}(\chi_\tilde{\sigma} F_\sigma)(\zeta)}{J^2(\zeta)}d\zeta,$$

where $\eta_\tilde{\sigma} = -I_\tilde{\sigma}$ and $z_\sigma \in \mathbb{R}^n + iI_\sigma$.

Let $\sigma \in \Lambda$ be fixed. Then for $\tilde{\sigma} \in \Lambda$ and $z_\sigma \in \mathbb{R}^n + iI_\sigma$, we have

$$S_{\sigma, \tilde{\sigma}}(z_\sigma) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iz_\sigma \zeta} \frac{\mathcal{F}(\chi_\tilde{\sigma} F_\sigma)(\zeta)}{J^2(\zeta)}d\zeta;$$

$$\left|S_{\sigma, \tilde{\sigma}}(z_\sigma)\right| \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-2\eta_\tilde{\sigma} \cdot y - z_\sigma \cdot \xi} \frac{\mathcal{F}(\chi_\tilde{\sigma} F_\sigma)(\zeta)}{J^2(\zeta)}d\xi.$$
One can see that $S_{\sigma,\tilde{\sigma}}(z_\sigma), z_\sigma \in \mathbb{R}^n + I_\sigma$ are continuable to the real axis. The corresponding functions $x \mapsto S_{\sigma,\tilde{\sigma}}(x)$ are continuous and of infra exponential type on $\mathbb{R}^n$. By Lemma 8.4.7 in [75], for $x \in \mathbb{R}^n$,

$$S_{\sigma,\tilde{\sigma}}(x) \cong S_{\sigma}(x + i\Gamma_\sigma 0), \quad \tilde{\sigma} \in \Lambda,$$

$$(f * g)(x) = \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} S_{\sigma,\tilde{\sigma}}(x), \quad x \in \mathbb{R}^n. \tag{1.23}$$

Functions $S_{\sigma,\tilde{\sigma}}$ can be written in the form

$$S_{\sigma,\tilde{\sigma}}(z_\sigma) = \frac{1}{(2\pi)^n} \sum_{\sigma \in \Lambda} \int_{\mathbb{R}^n} e^{iz_\sigma \zeta} F(\chi_\sigma F_\sigma)(\zeta_\tilde{\sigma})/J^2(\zeta_\tilde{\sigma}) \chi_\sigma(\zeta_\tilde{\sigma}) d\zeta, \quad z_\sigma \in \mathbb{R}^n + i I_\sigma.$$

Let $s \in \Lambda$. Then functions

$$S_{\sigma,\tilde{\sigma},s}(z_\sigma) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iz_\sigma \zeta} F(\chi_\sigma F_\sigma)(\zeta_\tilde{\sigma})/J^2(\zeta_\tilde{\sigma}) \chi_\sigma(\zeta_\tilde{\sigma}) d\zeta,$$

$z_\sigma \in \mathbb{R}^n + i I_\sigma, \quad \sigma, \tilde{\sigma}, s \in \Lambda$, are also continuable to the real axis. The corresponding functions $x \mapsto S_{\sigma,\tilde{\sigma},s}(x)$ are continuous and of infra exponential type on $\mathbb{R}^n$. Moreover, on $\mathbb{R}^n$,

$$S_{\sigma,\tilde{\sigma},s}(x) \cong S_{\sigma,\tilde{\sigma},s}(x + i\Gamma_\sigma 0),$$

$$S_{\sigma,\tilde{\sigma}}(x) = \sum_{s \in \Lambda} S_{\sigma,\tilde{\sigma},s}(x). \tag{1.24}$$

Let us analyze functions

$$I_{s,\varepsilon}(\zeta) = J^{-2}(\zeta)e^{-\varepsilon \imath \chi_\sigma(\zeta)}, \quad \zeta \in (\mathbb{R}^n + i\{||\eta|| < 1\}),$$

where $s \in \Lambda$ and $\varepsilon > 0$. These functions are elements of $\mathcal{P}_s$ because of

$$|I_{s,\varepsilon}(\zeta)| = |J^{-2}(\zeta)| \exp \left( -\varepsilon \sum_{i=1}^{n} s_i \zeta_i \right) \prod_{i=1}^{n} |\chi_\sigma(\zeta_i)| \leq |J^{-2}(\zeta)| \prod_{i=1}^{n} |\chi_\sigma(\zeta_i)| \exp(-\varepsilon \sum_{i=1}^{n} \zeta_i) \leq C \exp(\varepsilon \sum_{i=1}^{n} |\zeta_i|), \quad ||\eta|| < 1.$$

Therefore, $I_{s,\varepsilon} \in \tilde{\mathcal{O}}^{-\varepsilon}(\mathbb{D}^n + i\{||\eta|| < 1\}),\ s \in \Lambda$. Since the Fourier transform maps $\mathcal{P}_s$ onto $\mathcal{P}_s$, there exists $\psi_{s,\varepsilon} \in \mathcal{P}_s$ such that $F(\psi_{s,\varepsilon}) = I_{s,\varepsilon}$. By Proposition 8.2.2 in [75],

$$\psi_{s,\varepsilon} \in \tilde{\mathcal{O}}^{-1}(\mathbb{D}^n + i\{||\eta|| < \varepsilon\}), \quad s \in \Lambda.$$
1. S-asymptotics in $F'_g$

Denote by

$$q_s(x) = \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} S_{\sigma, \tilde{\sigma}, s}(x)$$

$$\cong \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} F^{-1}(F(F_\sigma \chi_{\tilde{\sigma}})J^{-2}(\chi_{\tilde{\sigma}}))(x + i(\Gamma_\sigma \cup \Gamma_s)0), x \in \mathbb{R}^n. \quad (1.25)$$

We prove that functions $q_s, s \in \Lambda$, have the properties cited in Theorem 1.11.

Property 1 follows from (1.24) and (1.25).

By (1.22) and (1.23) property 2 is satisfied.

It remains to prove property 3.

If $f \in \mathcal{Q}(D^n)$ and $\varphi \in \mathcal{P}_+$, then $f * \varphi \in \mathcal{O}(D^n + iI')$, where $I'$ is an interval containing zero. We shall use this fact and the properties of functions $I_{s, \epsilon}$, already analyzed.

For a fixed $s \in \Lambda$ there exists $\epsilon_0 > 0$ such that $\epsilon s$ belongs to all infinitesimal wedges of the form $D^n + i(\Gamma_\sigma \cup \Gamma_s)0$ which appear in (1.25). For $\epsilon \in (0, \epsilon_0]$, we have

$$q_s(x + i\epsilon s) = \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x + i\epsilon s) \zeta} F(F_\sigma \chi_{\tilde{\sigma}})(\zeta)J^{-2}(\chi_{\tilde{\sigma}})\chi_{s, \epsilon}(\zeta) d\zeta$$

$$= \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\epsilon s \zeta} F(F_\sigma \chi_{\tilde{\sigma}})(\zeta)F(\psi_{s, \epsilon})(\zeta) d\zeta$$

$$= \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} ((F_\sigma \chi_{\tilde{\sigma}}) * \psi_{s, \epsilon})(x)$$

$$= \left(\left(\sum_{\sigma \in \Lambda} F_\sigma\right) * \psi_{s, \epsilon}\right)(x) = (f * \psi_{s, \epsilon})(x).$$

Now, for every fixed $\epsilon \in (0, \epsilon_0]$, and $s \in \Lambda$,

$$\lim_{x \in \Gamma, \|x\| \to \infty} q_s(x + i\epsilon s)/c(x) = \lim_{x \in \Gamma, \|x\| \to \infty} (f * \psi_{s, \epsilon})(x)/c(x)$$

$$= \lim_{x \in \Gamma, \|x\| \to \infty} (f(t + x)/c(x), \psi_{s, \epsilon}(t)). \quad \square$$

We cite some papers related to this problem: ([131], [132], [155],[151], [123]).
1.10 S-asymptotic expansions in $\mathcal{F}_g'$

A sequence $\langle \psi_n \rangle_n$ of positive real-valued functions $\psi_n(t), t \in (t_0, \infty), t_0 \geq 0$ (defined on $(0, t_0), t_0 > 0$) is said to be asymptotic if and only if $\psi_{n+1}(t) = o(\psi_n(t)), \ t \to \infty (t \to 0)$. The formal series $\sum_{n \geq 1} u_n(t)$ is an asymptotic expansion of the function $u$ related to the asymptotic sequence $\{\psi_n(t)\}$ if

$$u(t) - \sum_{n=1}^{k} u_n(t) = o(\psi_k(t)), \ t \to \infty (t \to 0) \tag{1.26}$$

for every $k \in \mathbb{N}$. We write in this case:

$$u(t) \sim \sum_{n=1}^{\infty} u_n(t)\{\psi_n(t)\}, \ t \to \infty (t \to 0). \tag{1.27}$$

If $u_n(t) = c_n \psi_n(t)$, for every $n \in \mathbb{N}$, where $c_n$ are complex numbers, then expansion (1.27) is unique. Indeed, the numbers $c_n$ can be unambiguously computed from (1.26). In this case, we omit from the notation $\{\psi_n(t)\}$ in (1.27). A series which is an asymptotic expansion of a function $f$ can be also convergent. However, the series is divergent in general; nevertheless, several terms of it can give valuable information, and very often its good approximation properties come actually from the fact that the series is divergent. Sometimes, if we take more terms from the asymptotic series, we obtain a worse approximation; consequently, the determination of the optimal number of terms for a good approximation depends on a careful analysis of the problem under consideration.

An asymptotic expansion does not determine only one function. The following example illustrates this fact:

$$\exp \left( \frac{1}{x} \right) \sim \sum_{k=0}^{\infty} \frac{x^{-k}}{k!} \text{ and } \exp \left( \frac{1}{x} \right) + \exp(-x^2) \sim \sum_{k=0}^{\infty} \frac{x^{-k}}{k!}, \ x \to \infty. \tag{1.28}$$

In many problems of applied mathematics one is led to the use of asymptotic series. (See [44], [10], [15], [14], [192], [69]). A clear exposition of the theory and the use of asymptotic series of functions and distributions can be found in ([50]–[56]).

We shall discuss in this section the S-asymptotic expansion of generalized functions belonging to $\mathcal{F}_g'$. 
1. S-asymptotics in $F'_g$

1.10.1 General definitions and assertions

In this section $\Gamma$ will be a convex cone with the vertex at zero belonging to $\mathbb{R}^d$ and $\Sigma(\Gamma)$ the set of all real-valued and positive functions $c(h), h \in \Gamma$. We shall consider the asymptotic expansion when $\|h\| \to \infty, h \in \Gamma$.

**Definition 1.2.** A distribution $T \in F'_g$ has a S-asymptotic expansion related to the asymptotic sequence $\langle c_n(h) \rangle_n \subset \Sigma(\Gamma)$, if for every $\varphi \in F$

$$\langle T(t+h), \varphi(t) \rangle \sim \sum_{n=1}^{\infty} \langle U_n(t,h), \varphi(t) \rangle \{c_n(h)\}, \|h\| \to \infty, h \in \Gamma,$$

where $U_n(t,h) \in F'_g$ (with respect to $t$) for $n \in \mathbb{N}$ and $h \in \Gamma$. We write in short:

$$T(t+h) \sim \sum_{n=1}^{\infty} U_n(t,h) \{c_n(h)\}, \|h\| \to \infty, h \in \Gamma.$$

**Remarks.** 1) In the special case $U_n(t,h) = u_n(t)c_n(h), u_n \in F'_g, n \in \mathbb{N}$, we simply write

$$T(t+h) \sim \sum_{n=1}^{\infty} u_n(t)c_n(h), \|h\| \to \infty, h \in \Gamma.$$

In this case the given S-asymptotic expansion is unique.

2) Brychkov’s general definition is in $S'(\mathbb{R})$ and slightly different from ours ([15], [16] and [20]); his idea reformulated in $F'_g(\mathbb{R})$ gives the following definition. Suppose that $f \in F'_g(\mathbb{R})$ and that the function $\exp(ith)$, where $h$ is a real parameter, is a multiplier in $F'_g(\mathbb{R})$.

**Definition 1.3.** Suppose that $f \in F'_g(\mathbb{R})$. It is said that $f(x)e^{ixh}$ has an asymptotic expansion related to the asymptotic sequence $\langle \psi_n(h) \rangle_n$ if for every $\varphi \in F'_g(\mathbb{R})$

$$\langle f(x)e^{ixh}, \varphi(x) \rangle \sim \sum_{n=1}^{\infty} \langle C_n(x,h), \varphi(x) \rangle \{\psi_n(h)\}, h \to \infty,$$

where $C_n(x,h) \in F'_g(\mathbb{R})$ (with respect to $x$), $n \in \mathbb{N}, h \geq h_0$. We write in short:

$$f(x)e^{ixh} \sim \sum_{n=1}^{\infty} C_n(x,h) \{\psi_n(h)\}, h \to \infty.$$
To obtain an equivalent definition of this asymptotic expansion, Brychkov has supposed that \( F'(g(R)) = S'(R) \) and \( F(g(R)) = S(R) \). Then, by putting \( g = \hat{f} = F(f) \), \( c_n(\cdot,t) = F(C_n(\cdot,t)) \) and \( \phi = F(\varphi) \), Definition 1.3 reduces to:

**Definition 1.4.** A distribution \( g \in S'(R) \) has an asymptotic expansion related to the sequence \( \langle \psi_n(h) \rangle_n \) if for every \( \phi \in S(R) \)

\[
\langle g(h-t),\phi(t) \rangle \sim \sum_{n=1}^{\infty} \langle c_n(t,h),\phi(t) \rangle \{\psi_n(h)\}, h \to \infty,
\]

where \( c_n(\cdot,h) \in S'(R), n \in \mathbb{N} \) and \( h \geq h_0 \).

Definition 1.4 is, of course, a particular case of Definition 1.2 if we use \( T(t) = g(-t) \) and the cone \( \Gamma = R_- \).

In [15] and [20] authors studied asymptotic expansions of tempered distributions given by Definition 1.3. In [16] Brychkov extended Definition 1.4 to the \( n \)-dimensional case, but only on a ray \( \{ \varepsilon y; \varepsilon > 0 \} \) for a fixed \( y \in R^n \).

We study in this section the asymptotic expansion not only in \( S'(R) \) and not only on a ray, but on a cone in \( R^n \). The next remarks state some motivations for such investigations.

**Remarks.** 1) A distribution in \( S'(R) \) can have an S-asymptotic expansion in \( D'(R) \) without having the same S-asymptotic expansion in \( S'(R) \). Such an example is the regular distribution \( f \) defined by

\[
f(t) = H(t) \exp(1/(1 + t^2)) \exp(-t), \ t \in R,
\]

where \( H \) is the Heaviside function.

It is easy to prove that for \( h \in R_+ \)

\[
f(t+h) \sim \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (1+(t+h)^2)^{1-n} \exp(-t-h) \{ e^{-h} h^{2(1-n)} \}, h \to \infty,
\]

while

\[
t \mapsto U_n(t,h) = (1+(t+h)^2)^{1-n} \exp(-t-h), \ n \in \mathbb{N}, h > 0
\]

do not belong to \( S'(R) \).

2) The regular distribution \( g \) defined by

\[
g(t) = \exp(1/(1 + t^2)) \exp(t), \ t \in R
\]
1. S-asymptotics in $\mathcal{F}_g'$

belongs to $\mathcal{D}'(\mathbb{R})$ but not to $\mathcal{S}'(\mathbb{R})$. It has the S-asymptotic expansion in $\mathcal{D}'(\mathbb{R})$:

$$g(t + h) \sim \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (1 + (t + h)^2)^{1-n} \exp(t + h) \{e^{h} h^{2(1-n)}\}, \ h \to \infty,$$

where $\Gamma = \mathbb{R}_+$.

3) We can distinguish two cases of S-asymptotic expansions. If in Definition 1.2,

$$U_n(t, h) = u_n(t) c_n(h), \ n \in \mathbb{N},$$

then the S-asymptotic expansion is called of second type. (See Remark after Definition 1.2). If $U_n(t, h) = u_n(t + h), \ n \in \mathbb{N}$, then the S-asymptotic expansion is of first type.

The following example illustrates the difference between these two types of S-asymptotic expansions.

Let $f(x) = \sqrt{x^2 + x}, \ x > 0$ and $f(x) = 0, \ x \leq 0$. A S-asymptotic expansion of the first type for this distribution is

$$f(x + h) \sim \sum_{n=1}^{\infty} \left(\frac{1/2}{n - 1}\right)(x + h)^{2-n} \{h^{2-n}\}, \ h \to \infty,$$

but the sequence $u_n(x) = \left(\frac{1/2}{n - 1}\right)x^{2-n}$ cannot give a S-asymptotic expansion of the second type for $f$.

S-asymptotic expansions have similar properties as those of S-asymptotics.

**Theorem 1.12.** Let $T \in \mathcal{F}_g'$ and

$$T(t + h) \sim \sum_{n=1}^{\infty} U_n(t, h) [c_n(h)], \ \|h\| \to \infty, \ h \in \Gamma.$$

Then:

a) $T^{(k)}(t + h) \sim \sum_{n=1}^{\infty} U_n^{(k)}(t, h) [c_n(h)], \ \|h\| \to \infty, \ h \in \Gamma.$
b) Let the open set $\Omega$ have the property: for every $r > 0$ there exists a $\beta_0$ such that the closed ball $B(0, r) = \{x \in \mathbb{R}^n, \|x\| \leq r\}$ is in $\{\Omega - h, h \in \Gamma, \|h\| \geq \beta_0\}$. If $T, T_1 \in F'_0$ and $T_1 = T$ over $\Omega$, then

$$T_1(t + h) \overset{\sim}{\to} \sum_{n=1}^{\infty} U_n(t, h)[\{c_n(h)\}], \|h\| \to \infty, h \in \Gamma,$$

as well.

c) Assume additionally that $F_g$ is a Montel space and that in $F_g$ the convolution is well defined (see 0.6) and hypocontinuous. Let $S \in F'_g$, supp $S$ being compact. Then

$$(S \ast T)(t + h) \overset{\sim}{\to} \sum_{n=1}^{\infty} (S \ast U_n)(t, h)[\{c_n(h)\}], \|h\| \to \infty, h \in \Gamma.$$

Proof. We prove only a). The proofs of b) and c) are the same as in the proof of Theorem 1.2. We have

$$\lim_{h \in \Gamma, \|h\| \to \infty} \frac{\langle T^{(k)}(t + h) - \sum_{n=1}^{m} U^{(k)}_n(t, h), \varphi(t) \rangle}{c_m(h)} = 0. \quad \square$$

A relation between the asymptotic expansion of a locally integrable function $f$ and its S-asymptotic expansion when seen as a regular generalized function is provided in the following proposition (see [152]).

**Proposition 1.8.** Let $f(t), U_n(t, h)$ and $V_n(t), t \in \mathbb{R}^n, n \in \mathbb{N}, h \in \Gamma$, be locally integrable functions such that for every compact set $K \subset \mathbb{R}^n$ the following ordinary asymptotic expansion holds,

$$f(t + h) \sim \sum_{n=1}^{\infty} U_n(t, h)[\{c_n(h)\}], \|h\| \to \infty, h \in \Gamma, t \in K$$

and for every $k \in \mathbb{N}$

$$\frac{1}{c_k(h)} \left| f(t + h) - \sum_{n=1}^{k} U_n(t, h) \right| \leq V_k(t), \ t \in K, h \in \Gamma, \|h\| \geq r(k, K).$$

Then for $f \in F'_0$, we have

$$f(t + h) \overset{\sim}{\to} \sum_{n=1}^{\infty} U_n(t, h)[\{c_n(h)\}], \|h\| \to \infty, h \in \Gamma.$$
Proposition 1.9. Suppose that $\Gamma$ has the nonempty interior. Let $T \in \mathcal{F}'$ and

$$T(t + h) \sim \sum_{n=1}^{\infty} u_n(t) c_n(h), \quad \|h\| \to \infty, \quad h \in \Gamma.$$ 

If $u_m \neq 0$, $m \in \mathbb{N}$, then $u_m$ has the form

$$u_m(t) = \sum_{k=1}^{m} P_k^m(t) \exp(a^k \cdot t), \quad t \in \mathbb{R}^n, \quad m \in \mathbb{N},$$

where $a^k = (a^k_1, \ldots, a^k_n) \in \mathbb{R}^n$ and $P_k^m$ are polynomials with degrees less than $k$ at every $t_i$, $i = 1, \ldots, n$.

Proof. Definition 1.2 and the given asymptotics implies

$$\lim_{h \in \Gamma, \|h\| \to \infty} T(t + h)/c_1(h) = u_1(t) \neq 0 \quad \text{in} \quad \mathcal{F}'.$$

Now Proposition 1.2 implies the explicit form of $u_1$.

The following limit gives $u_2$:

$$\lim_{h \in \Gamma, \|h\| \to \infty} \frac{(T(t + h), \varphi(t)) - (u_1(t), \varphi(t))c_1(h)}{c_2(h)} = (u_2, \varphi), \quad \varphi \in \mathcal{F}.$$ 

Note,

$$\lim_{h \in \Gamma, \|h\| \to \infty} \frac{\langle (D_{t_i} - a_i^1)T(t + h), \varphi(t) \rangle}{c_2(h)} = \langle (D_{t_i} - a_i^1)u_2(t), \varphi(t) \rangle, \quad \varphi \in \mathcal{F}.$$

Two cases are possible.

a) If $(D_{t_i} - a_i^1)u_2 = 0, \quad i = 1, \ldots, n$, then $u_2(t) = C_2 \exp(a^1 \cdot t)$.

b) If $(D_{t_i} - a_i^1)u_2 \neq 0$ for some $i$, then by Proposition 1.2, $(D_{t_i} - a_i^1)u_2(t) = c \exp(a^2 \cdot t)$ and $u_2$ has the form

$$C_2 \exp(a^1 \cdot t) + P_2^2(t_1, \ldots, t_n) \exp(a^2 \cdot t),$$

where $P_2^2$ is a polynomial of the degree less than 2 with respect to each $t_i, \quad i = 1, \ldots, n$.

In the same way, we prove the assertion for every $u_m$. \hfill \Box

We will give an example of a function which has S-asymptotic expansion of the first type but does not have the asymptotic expansion as a function:

Let $\psi(t) = 1$, $t \in (n - 2^{-n}, n + 2^{-n}), \quad n \in \mathbb{N}$, and $\psi(t) = 0$ outside of these intervals. Let

$$\psi_a(x) = e^{ax} \int_0^x \psi(t)dt, \quad x \in \mathbb{R}, \quad a \in \mathbb{R}.$$
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Since \( \int_0^x \psi(t) dt \to 2 \) as \( x \to \infty \), we have that \( \psi(x) \sim 2e^{\alpha x}, x \to \infty \) but \( \psi'(x) \) does not have an ordinary asymptotic behavior (see Example 5 in 1.5.1).

Let \( \langle \alpha_j \rangle_j \) be a strictly decreasing sequence of positive numbers. Let \( \theta \) be a function, \( \theta \in C^\infty, \theta \equiv 1 \) for \( x > 1, \theta \equiv 0 \) for \( x < 1/2 \) and \( f(x) = \sum_{i=1}^\infty \psi_{\alpha_i}(x) \theta(x-i), x \in \mathbb{R} \). We have
\[
f(x) \sim \sum_{i=1}^\infty \psi_{\alpha_i}(x) \{ 2 \exp(\alpha_i x) \}, \quad x \to \infty.
\]
This implies that \( f \) has the S-asymptotic expansion of the first type:
\[
f(x + h) \sim \sum_{i=1}^\infty \psi_{\alpha_i}(x + h) \{ 2 e^{\alpha_i h} \}, \quad h \to \infty
\]
and
\[
F(x + h) = f'(x + h) \sim \sum_{i=1}^\infty \psi'_{\alpha_i}(x + h) \{ 2 e^{\alpha_i h} \}, \quad h \to \infty
\]
but \( F \) does not have the ordinary asymptotic expansion.

For S-asymptotic expansions see also ([108], [152]).

1.10.2 S-asymptotic Taylor expansion

Estrada and Kanwal have introduced in [53] and [54] the asymptotic Taylor expansion of distributions (see also [56]). We generalize it to \( F'_g \).

Definition 1.5. Let \( f \in F'_g \) and \( D^k f \) be the \( k \)-th partial derivative of \( f \). A formal series
\[
\sum_{|k|=0}^\infty \frac{D^k f(x) y^k}{k!} \varepsilon^{|k|}, \quad \text{where } y \in \mathbb{R}^n \text{ is fixed},
\]
is the asymptotic Taylor expansion of \( f \), as \( \varepsilon \to 0 \).

It means that for any test function \( \phi \in F_g \)
\[
\langle f(x + \varepsilon y), \phi(x) \rangle = \sum_{|k|=0}^{N} \frac{(D^k f(x), \phi(x)) y^k}{k!} \varepsilon^{|k|} = O(\varepsilon^{N+1}), \quad \text{as } \varepsilon \to 0.
\]
We write in short
\[ f(x + εy) \sim \sum_{|k| = 0}^{∞} \frac{(D^k f(x)y^k)}{k!} ε^{|k|}, \quad ε \to 0. \tag{1.31} \]

In fact, the asymptotic Taylor expansion is the S-asymptotic expansion related to the asymptotic sequence \( \{c_n(h) = h^n; n \in \mathbb{N}\} \), where \( h \to 0 \), with \( U_n(x, h) = u_n(x)c_n(h) \) and \( Γ = \{ε y; \ ε > 0\} \), where \( y \in \mathbb{R}^n \) is fixed (compare with Definition 1.2). We illustrate Definition 1.5 by two examples:

1. \( \delta(x + εy) \sim \sum_{|N| = 0}^{∞} \frac{1}{N!}(D^N \delta(x)y^N)ε^N, \quad as \ ε \to 0, \)
   where \( ε^N = ε^{N_1+⋯+N_n} \).

2. If \( α \not\in \mathbb{Z} \), then the distribution \( x_+^α \in \mathcal{D}'(\mathbb{R}) \) has the asymptotic Taylor expansion
   \[ (x - ε)_+^α \sim \sum_{k=0}^{∞} \binom{α}{k} (-1)^k x_+^{α-k} ε^k, \quad ε \to 0, \]
   This means that if \( φ \in \mathcal{D}(\mathbb{R}) \), then \( F.p. \int_0^{∞} (x - ε)_+^α φ(x)dx \sim \sum_{k=0}^{∞} \binom{α}{k} (-1)^k F.p. \int_0^{∞} x^{α-k} φ(x)dx \ ε^k, \quad ε \to 0, \)
   where F.p. stands for the finite part (cf. [146], [56]).

We now discuss the problem of the convergence of an asymptotic Taylor expansion.

If a regular distribution is defined by a real analytic function \( f \), then the asymptotic Taylor expansion for \( f \) is a convergent power series in an appropriate domain. But if \( f \) is only smooth, then the asymptotic Taylor expansion for \( f \) does not necessarily have a domain of convergence. The asymptotic Taylor expansion of \( δ \) distribution, given in Example 1, is not convergent in \( \mathcal{D}'(\mathbb{R}) \); we know that the series \( Σ a_k δ^{(k)} \) diverges unless \( a_k = 0, \|k\| ≥ n_0 \in \mathbb{N} \).

One could attempt to solve this problem by trying to interpret the asymptotic Taylor expansion for a distribution in a wider space \( \mathcal{F}_g' \) of generalized functions asking the question: Find a necessary and sufficient condition for the asymptotic Taylor expansion of a generalized function to be convergent in \( \mathcal{F}_g' \), i.e., so that it becomes its Taylor series.

In the following theorems we give the answer to this question if \( \mathcal{F}_g' \) is the space of distributions \( \mathcal{F}_g' = \mathcal{D}' \), the space of ultradistributions in which
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(M.1), (M.2) and (M.3) are satisfied (see 0.5.2) \( F_g \) and the space of Fourier hyperfunctions \( F_g = \mathcal{Q} \).

**Theorem 1.13.** ([163]). Let \( f \in D' (f \in D'^*) \) and \( y = (y_1, \ldots, y_n) \) \( y_i \neq 0 \), \( i = 1, \ldots, n \). The asymptotic Taylor expansion for \( f \), on the straight line \( \{ y; \varepsilon \in \mathbb{R} \} \), is a convergent Taylor series in \( D' \) (in \( D'^* \)) if and only if there exists an \( r = (r_1, \ldots, r_n) \), \( r_i > 0 \), \( i = 1, \ldots, n \), such that \( f \) is a real analytic function on \( \mathbb{C}^n; |\text{Im} \ z_i| < r_i, i = 1, \ldots, n \).

**Proof.** Suppose that \( f \in D' (f \in D'^*) \) and that \( w \in D \ (w \in D^*) \). Then

\[
(f(x + \varepsilon y), w(x)) = (f \ast \tilde{w})(\varepsilon y), \ y \in \mathbb{R}^n, \varepsilon \in \mathbb{R}
\]

and

\[
(f \ast \tilde{w})(\varepsilon y + y_0) = (f(t) \ast \tilde{w}(t - y_0))(\varepsilon y), \ y, y_0 \in \mathbb{R}^n,
\]

where \( \ast \) is the sign of convolution and \( \tilde{w}(t - y_0) = w(-t + y_0) \). This shows that the expansion in a polydisc around \( y_0 \) can be transferred to the expansion around 0.

In the sequel we shall use the following results proved in ([146], Chap. VI, Theorem XXIV and [161], Theorem 1):

*The generalized function \( f \in D' (f \in D'^*) \) is a real analytic function if and only if \( f \ast w \) is real analytic for every \( w \in D \ (w \in D^*) \).*

\( \Leftarrow \) Suppose that there exists \( r = (r_1, \ldots, r_n) \), \( r_i > 0 \), \( i = 1, \ldots, n \) such that \( f \) is holomorphic in \( \{x \in \mathbb{C}^n; |\text{Im} z_i| < r_i, i = 1, \ldots, n \} \). Then its Taylor series converges in \( D(a, r) = D(a_1, r_1) \times \cdots \times D(a_n, r_n) \) for every \( a \in \mathbb{R}^n \). Therefore for every \( r' = (r'_1, \ldots, r'_n), \ 0 < r'_i < r_i, \ i = 1, \ldots, n \) and every compact set \( K \subset \mathbb{R}^n \), by the characterization of a real analytic function,

\[
|f^{(k)}(x)| \leq C(r')^{-k} k!, \ k \in \mathbb{N}_0^n, \ x \in K.
\]

Let us prove that

\[
\sum_{|k|=0}^{\infty} \frac{f^{(k)}(x)}{k!} (r')^k
\]
converges in $\mathcal{D}'$ (in $\mathcal{D}'*$) for any $r''$, $0 < r''_i < r'_i$, $i = 1, \ldots, n$. It is enough to prove that for every $w \in \mathcal{D}$ ($w \in \mathcal{D}'$) and every $p \in \mathbb{N}$

$$\lim_{N \to \infty} \left( \sum_{|k|=N} \frac{f(k)(x)}{k!} (r''_i)^k, w(x) \right)$$

$$= \lim_{N \to \infty} \sum_{|k|=N} \int_{\mathbb{R}^n} (f(k)(x)/(k!)) w(x) dx (r''_i)^k = 0.$$ Since for every $w \in \mathcal{D}$ ($w \in \mathcal{D}'$)

$$\left| \int_{\mathbb{R}^n} (f(k)(x)/(k!)) w(x) dx \right| \leq C(r')^{-k} \int_{\mathbb{R}^n} |w(x)| dx, \quad |k| \geq n_0(K),$$

the series (1.34) converges for every $r''$, $0 < r''_i < r'_i$. Consequently, there exists $\varepsilon \neq 0$ such that $0 < |\varepsilon y_i| < r_i$ and that series (1.31) converges in $\mathcal{D}'$ (in $\mathcal{D}'*$).

$\Rightarrow$ Suppose now that the series (1.31) converges in $\mathcal{D}'$ (in $\mathcal{D}'*$) for a fixed $y \in \mathbb{R}^n$ and a fixed $\varepsilon > 0$. Then for every $w \in \mathcal{D}$ ($w \in \mathcal{D}'$)

$$\sum_{|k|=0}^{\infty} \frac{D^k f(x)}{k!} (y \varepsilon)^k$$

is a convergent series in the polydisc $D(0, r)$, where $r_i = y_i \varepsilon, i = 1, \ldots, n$. By (1.33), $(f \ast w)(z)$ is a holomorphic function on $\{z \in \mathbb{C}^n; |\text{Im} z_i| < r_i, i = 1, \ldots, n\}$. By the cited theorem, $f$ is a real analytic function. We now continue the proof dividing it into two cases.

**Case $f \in \mathcal{D}'$.**

By (VI, 6; 22) in [146]

$$f = \Delta^k (gE \ast f) - (v \ast f), \quad (1.35)$$

where $\Delta$ is the Laplacian, $v \in \mathcal{D}$, and $k$ is large enough so that for a fixed $m, gE \in \mathcal{D}_V^m$, where $V$ is a relatively compact neighborhood of zero.

By the property of $f \ast w$ we just proved, it follows that for every $w \in \mathcal{D}$ and for every $y_0 \in \mathbb{R}^n$ there exists $M > 0$ such that

$$(r''_i)^k |(f(k) \ast \omega)(y_0)| \leq M k!,$$

where $k \in \mathbb{N}_0^n$ and $0 < r'' < r$. By Theorem XXII Chap. VI in [146], there exists $m \geq 0$ such that this inequality is also true if $w = \bar{w} \in \mathcal{D}_V^m$, where $V$
is a relatively compact neighborhood of zero. Consequently, it follows that \( f \star \tilde{w} \) is a holomorphic function in \( B(y_0, r) \). By (1.35), \( f \) is also holomorphic in \( B(y_0, r) \) for every \( y_0 \in \mathbb{R}^n \).

**Case** \( f \in D' \). In the proof of this case we use the following Theorem ([81]):

Let the sequence \( M_p \) satisfy conditions (M1), (M2), and (M3). For a given \( H_p \) and a compact neighborhood \( Q \) of zero in \( \mathbb{R}^n \) there exist an ultradifferential operator \( P(D) \) of class * and functions \( \varphi \in C^\infty \), and \( w \in D^* \) such that

\[
P(D)\varphi = \delta + w
\]

\[
\text{supp} \varphi \subset Q, \quad \sup_{x \in Q} |\varphi^{(k)}(x)|/H_{|k|}M_{|k|} \to 0, \quad |k| \to \infty.
\]

Following the proof of Theorem 1 in [161], one can conclude that if

\[
|(D^k(f \star w))(y_0)| = |(f \star w)^{(k)}(y_0)| \leq C k! (r')^{-k},
\]

where \( w \in D^* \) and \( k \in \mathbb{N}_0 \), then there exists \( H_p \) such that the same inequality holds for \( w = \tilde{w} \in \tilde{D}^{H_p,M_p} \), where \( \tilde{D}^{H_p,M_p} \) is the completion of \( D^* \) under the norm \( g_{H_p,M_p} \). Therefore \( f \star \varphi \) is a holomorphic function in \( B(y_0, r) \).

By the cited theorem

\[
f = P(D)(\varphi \star f) + f \star w, \tag{1.36}
\]

where \( \varphi \star f \) and \( f \star w \) are holomorphic in a ball around any point of \( \mathbb{R}^n \).

So the proof will be finished if we prove that for a real analytic function \( \theta \) and an ultradifferential operator \( P(D) \) of (*)-class, \( P(D)\theta \) is real analytic.

Therefore, we will prove first the next assertion.

**Lemma 1.3.** Let \( P(D) \) be an ultradifferential operator of (*)-class and \( \theta \) be real analytic in a neighborhood of \( x_0 \in \mathbb{R}^n \). Then \( P(D)\theta \) is real analytic in a neighborhood of \( x_0 \).

**Proof.** We will prove the assertion in the case \( n = 1 \) and \( P(D) \) being of \( \{M_p\} \)-class which is equivalent to \( P(D) = \Sigma a_k D^k \), where \( a_k \in \mathbb{C}, \quad k \in \mathbb{N}_0 \) and there exist \( C > 0 \) and \( h > 0 \) such that

\[
|a_k| \leq \frac{C h^k}{M_k}, \quad k \in \mathbb{N}_0.
\]
We have to prove that there exists $H > 0$ such that in a ball $B_r = \{ x : |x - x_0| \leq r \}$

$$\sup_{\alpha \in \mathbb{N}_0, x \in B_r} \frac{H^\alpha |(P(D)\theta)^{(\alpha)}(x)|}{M_\alpha} < \infty.$$ 

We have $(x \in B_r, \alpha \in \mathbb{N}_0)$

$$\frac{1}{M_\alpha} \left| H^\alpha D^\alpha \sum_{k=0}^{\infty} a_k D^k \theta(x) \right| \leq \frac{H^\alpha}{M_\alpha} \sum_{k=0}^{\infty} a_k D^{\alpha+k} \theta(x) \leq \left( C_1 H^\alpha \sum_{k=0}^{\infty} \frac{h^k}{M_k} C_1^{k+1} (k+\alpha)! \right),$$

where we have used $|D^k \theta(x)| \leq C_1^{k+1} k!, x \in B_r$ which holds for some $C_1 > 0$. Since $(k + \alpha)! \leq e^{k+\alpha} k! \alpha!$, we continue

$$\leq CC_1 \sum_{k=0}^{\infty} e^{\alpha+k} \frac{H^\alpha h^k C_1^{k+1} k!}{M_\alpha M_k} \leq C \sup_\alpha \frac{(C_1 e)^{\alpha+1} H^\alpha}{M_\alpha} \sum_{k=0}^{\infty} \frac{(C_1 h)^k k!}{M_k}.$$ 

The right side is bounded for every $H > 0$. This proves the lemma and consequently the Theorem. \hfill \Box

**Theorem 1.14.** ([164]) Let $q \in \mathcal{Q}$, $\xi = (\xi_1, \ldots, \xi_n)$, $\xi_i > 0$, $i = 1, \ldots, n$ and $\varepsilon > 0$. The asymptotic Taylor expansion (1.31) for $q$ on the straight line $\{ h \xi; h \in \mathbb{R} \}$ is a convergent Taylor series in the topology of $\mathcal{Q}$ if and only if there exists an $r = (r_1, \ldots, r_n)$, $r_i > 0$, $i = 1, \ldots, n$, such that $q$ is given by a real analytic function which can be extended as a holomorphic function on $\{ z \in \mathbb{C}^n; |\text{Im } z_i| < r_i, i = 1, \ldots, n \}$.

**Proof.** In the first part of the proof, we suppose that series (1.31) converges in $\mathcal{Q}$ for a fixed $\varepsilon_1 = (\xi_1, \ldots, \xi_n)$, $\xi_i > 0$, $i = 1, \ldots, n$, and a fixed $\varepsilon > 0$. Since $\mathcal{Q}$ is an FS-space it is equivalent to suppose that series (1.31) converges weakly in $\mathcal{Q}$.

By Theorem 1.3 and Remark 1.4 in [73], there exists an elliptic local operator $J_1(D)$ and an infinitely differentiable function $g$ rapidly decreasing ($|g(x)| \leq C \exp(-\alpha \|x\|)$, $x \in \mathbb{R}^n$ for some $\alpha > 0$) such that

$$\delta = J_1(D)g \ (\delta \text{ is the delta distribution}).$$

Also (cf. Theorem in [74]) for every Fourier hyperfunction $q \in \mathcal{Q}$, we can find an elliptic local operator $J_2(D)$ and an infinitely differentiable
function $f$ of infra-exponential growth such that $q = J_2(D)f$. Then, by the properties of the convolution:

$$q = (J_2(D)f) * (J_1(D)g) = J_2(D)J_1(D)((\ell f) * (\ell g)), \quad (1.37)$$

where $J(D) = J_2(D)J_1(D)$ is also an elliptic operator. $(\ell f$ denotes the hyperfunction defined by the function $f$).

From the proof of two mentioned theorems and by Theorem 8.2.6 in [75] it follows that there exist two sets of functions $\{F_\sigma; \sigma \in \Lambda\}$ and $\{G_{\tilde{\sigma}}; \tilde{\sigma} \in \Lambda\}$ such that:

a) $F_\sigma \in \tilde{\mathcal{O}}(D^n + i\Gamma_\sigma 0)$ and $G_{\tilde{\sigma}} \in \tilde{\mathcal{O}}^{-\alpha}(D^n + i\Gamma_{\tilde{\sigma}} 0)$, $\alpha > 0$, $\sigma, \tilde{\sigma} \in \Lambda$;

b) the functions $F_\sigma$ and $G_{\tilde{\sigma}}$ can be extended to the real axis $\mathbb{R}^n$ as infinitely differentiable functions $F_\sigma(x)$ and $G_{\tilde{\sigma}}(x)$, respectively. $F_\sigma(x)$ are infra-exponential and $G_{\tilde{\sigma}}(x)$ are rapidly decreasing, $\sigma, \tilde{\sigma} \in \Lambda$.

c) $f(x) = \sum_{\sigma \in \Lambda} F_\sigma(x)$ and $g(x) = \sum_{\tilde{\sigma} \in \Lambda} G_{\tilde{\sigma}}(x)$, $x \in \mathbb{R}^n$; $g$ is analytic outside $\{0\}$.

By Carleman's theorem, we have

$$f(x) := \sum_{\sigma \in \Lambda} F_\sigma(x + i\Gamma_\sigma 0), \quad g(x) := \sum_{\tilde{\sigma} \in \Lambda} G_{\tilde{\sigma}}(x + i\Gamma_{\tilde{\sigma}} 0), \quad x \in \mathbb{R}^n$$

(cf. [75] Lemma 8.4.7, and [145], §7).

Our first step is to prove that the Fourier hyperfunction $(\ell f) * (\ell g)$ is defined by the infra-exponential function

$$\int_{\mathbb{R}^n} f(x-u)g(u)du, \quad x \in \mathbb{R}^n.$$

By the definition of the convolution, we have

$$(\ell f) * (\ell g)(u) := \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} \int_{\mathcal{M}_w=v_\sigma} F_\sigma(z-w)G_{\tilde{\sigma}}(w)dw, \quad w = u + iv, \quad v_\sigma \in I_{\sigma}, \quad (1.38)$$

where $z - w \in \mathbb{R}^n + i(\Gamma_\sigma + \Gamma_{\tilde{\sigma}})0$. Hence $z$ can move inside $\mathbb{R}^n + i(\Gamma_\sigma + \Gamma_{\tilde{\sigma}})0$. We can shift the integral path in the last integral to the real axis $\mathbb{R}^n$. This change of the path is justified by Cauchy's integral formula and the growth rate of the functions $f$ and $g$ at infinity. Then we have

$$(\ell f) * (\ell g)(x) := \sum_{\sigma} \sum_{\tilde{\sigma}} \int_{\mathbb{R}^n} F_\sigma(z-u)G_{\tilde{\sigma}}(u)du.$$
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Every integral in the last sum is a function into $\tilde{O}(D^n + i(\Gamma_\sigma + \tilde{\Gamma}_\sigma)0)$, respectively, and can be extended to the real axis as a slowly increasing continuous function. Therefore by the same Carlemann’s theorem

$$\ell(f) * (\ell g)(x) = \ell \left( \sum_{\sigma} \sum_{\tilde{\sigma}} \int_{\mathbb{R}^n} F_\sigma(x-u) G_{\tilde{\sigma}}(u) du \right)$$

$$= \ell \left( \int_{\mathbb{R}^n} f(x-u) g(u) du \right).$$

(1.39)

Hence, the first step is proved.

In the second step, we shall prove that $(\ell f) * (\ell g)(x)$ is a real analytic function in a neighborhood of zero, that is it can be extended on a complex neighborhood of zero as a holomorphic function.

By definition of an infinitesimal wedge $U_{\sigma_1}$ of type $D^n + \Gamma_{\sigma_1}0$ for every proper subcone $\Gamma'_{\sigma_1} \subset \Gamma_{\sigma_1}$ there exists $w > 0$ such that $U_{\sigma_1} \supset D^n + i(\Gamma'_{\sigma_1} \cap \{|y| < w\})$. Put $\tilde{G}(u) = G(-u + iw/2)$. Then $\tilde{G} \in \tilde{O}^{-\alpha}(D^n + i\{|v| < w/2\})$ and $\tilde{G} \in \mathcal{P}_*$, as well.

Let us consider an addend in (1.38) and the corresponding integral in (1.39). We have

$$\ell(F_\sigma * G_{\tilde{\sigma}})(x) := \int_{\mathbb{R}^n} F_\sigma(z - \left(u + \frac{1}{2} iv_{\tilde{\sigma}}\right)) G_{\tilde{\sigma}}(u + \frac{1}{2} iv_{\tilde{\sigma}}) du, \quad x \in \mathbb{R}^n,$$

where $v_{\tilde{\sigma}} \in \Gamma'_{\sigma} \cap \{|y| < \frac{1}{2} w\}$. Thus, for $x \in \mathbb{R}^n$,

$$(F_\sigma * G_{\tilde{\sigma}}) \left(x + \frac{1}{2} iv_{\tilde{\sigma}}\right) = \int_{\mathbb{R}^n} F_\sigma(x-u) \tilde{G}_{\tilde{\sigma}}(-u) du$$

$$= \int_{\mathbb{R}^n} F_\sigma(x+u) \tilde{G}_{\tilde{\sigma}}(u) du$$

$$= \int_{\mathbb{R}^n} F_\sigma(x+u + \frac{1}{2} iv_{\tilde{\sigma}}) \tilde{G}_{\tilde{\sigma}}(u + \frac{1}{2} iv_{\tilde{\sigma}}) du.\quad (1.40)$$

By (1.38)–(1.40), we have

$$((\ell f) * (\ell g)) \left(x + \frac{1}{2} iv_{\tilde{\sigma}}\right) = \sum_\tilde{\sigma} (\ell f(x+u), \tilde{G}_{\tilde{\sigma}}(u)), \quad \tilde{G}_{\tilde{\sigma}} \in \mathcal{P}_*.$$

Let $\varepsilon > 0$ be fixed. The assumption of the theorem, implies that $(q(x+u), G_{\tilde{\sigma}}(u)), \tilde{G}_{\tilde{\sigma}} \in \mathcal{P}_*$, as a function in $x \in \mathbb{R}^n$, can be extended
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on $B(0, \varepsilon \xi) \subset \mathbb{C}^n$ as a holomorphic function. Then
\[ \langle q(x + u), G_{\sigma}(u) \rangle = \langle J_2(D)f(x + u), G_{\sigma}(u) \rangle \]
\[ = J_2(D)\langle f(x + u), G_{\sigma}(u) \rangle, \quad x \in \mathbb{R}^n. \]

Since $J_2(D)$ is an elliptic local operator, it follows that for every $\tilde{\sigma} \in \Lambda$, $\langle f(\cdot + u), G_{\tilde{\sigma}}(u) \rangle$ is also real analytic. Consequently, $\langle (\ell f) \ast (\ell g) \rangle$ is holomorphic in $B(0, \beta)$ for $\beta, \beta_i > 0$, $i = 1, \ldots, n$.

Now it is easy to prove that $\langle (\ell f) \ast (\ell g) \rangle(z)$ is holomorphic on $\mathbb{R}^n + B(0, \beta)$.

In order to finish the first part of the proof, we have only to remark that from (1.37) and from the property of an elliptic local operator it follows that $q$ is a real analytic function which can also be extended as a holomorphic function on $\mathbb{R}^n + B(0, \beta)$.

For the second part of the proof suppose that $q$ is given by a real analytic function which can be extended on $\mathbb{R}^n + B(0, \beta)$, $\beta_i > 0$, $i = 1, \ldots, n$, as a holomorphic function. Then for $\varepsilon > 0$ and $\xi \in \mathbb{R}^n$,
\[ \ell q(\varepsilon \xi + x) := q(\varepsilon \xi + x + iI_{\sigma_0}), \quad x \in \mathbb{R}^n \]
and, for $y_0 \in \text{Im} \sigma_0$,
\[ \langle \ell q(\varepsilon \xi + x), \varphi(x) \rangle = \int_{1mz = y_0} q(\varepsilon \xi + x + iy_0)\varphi(x + iy_0)dx, \quad \varphi \in \mathcal{P}_*. \]

By the same arguments which we have used to transform (1.38) into (1.39), we have
\[ \langle \ell q(\varepsilon \xi + x), \varphi(x) \rangle = \int_{\mathbb{R}^n} q(\varepsilon \xi + x)\varphi(x)dx, \quad \varphi \in \mathcal{P}_*. \]

Since $q(z)$ is holomorphic on $\mathbb{R}^n + B(0, \beta)$, the same holds for $\langle \ell q(\varepsilon \xi + x), \varphi(x) \rangle$. \qed

For asymptotic Taylor expansions, see also [163], [1], [50], [56].

1.11 S-asymptotics in subspaces of distributions

We discuss in this section the following problem: Let $\mathcal{A}'$ be a subspace of $\mathcal{D}'$. If $T \in \mathcal{A}'$ and if $T$ has the S-asymptotics in $\mathcal{D}'$, is it true that $T$ has the S-asymptotics in $\mathcal{A}'$, as well? The answer is not simple. We shall analyze two cases, $\mathcal{A}' = S'$ and $\mathcal{A}' = \mathcal{K}'_1$. First, we shall illustrate the problem by an example.
Proposition 1.10. Suppose that $c \in \mathbb{R}$. It has S-asymptotics in $\mathcal{D}'(\mathbb{R})$ related to $c(h) = \exp(-h)$ with limit $U = \exp(-t)$. We know that $\exp(-t)$ does not belong to $\mathcal{S}'(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$ is dense in $\mathcal{S}(\mathbb{R})$. Hence $H(t)\exp(-t)$ cannot have the S-asymptotics in $\mathcal{S}'(\mathbb{R})$.

To answer our question, we need to introduce some additional conditions over $c(h)$.

Although we defined the S-asymptotics of distributions in a general cone $\Gamma$, we shall restrict ourselves to the cone $\Gamma = \mathbb{R}^n_+$ for the sake of simplicity. Recall, for $a, b \in \mathbb{R}^n$, $a \geq b$ means $a_i \geq b_i, i = 1, \ldots, n$.

The set of real-valued functions $h \rightarrow c(h)$, $h \in \mathbb{R}^n$, defined on $\mathbb{R}^n$, different from zero for $h \in \mathbb{R}^n_+$, is denoted by $\Sigma(\mathbb{R}^n)$. We assume, without losing generality, that function $c$ in $\Sigma(\mathbb{R}^n)$ are positive and equal to 1 in $\mathbb{R}^n_+ \setminus (\mathbb{R}^n_+ + a)$, where $a \in \mathbb{R}^n_+$ depends on $c$ and $\mathbb{R}^n_+ + a = \{x + a; x \in \mathbb{R}^n_+\}$.

By $\Sigma_c(\mathbb{R}^n)$ is denoted the subset of $\Sigma(\mathbb{R}^n)$ such that $c \in \Sigma_c(\mathbb{R}^n)$ if and only if for some $C > 0, d > 0, k > 0$ and $h_0 = (h_{0,1}, \ldots, h_{0,n}) \in \mathbb{R}^n_+$:

\begin{equation}
(E.1)
\quad c(h + r) \leq Cc(h)\exp(k\|r\|), \quad h > h_0, \quad r \in \mathbb{R}^n.
\end{equation}

\begin{equation}
(E.2)
\quad c(h)\exp(k\|h\|) \geq d, \quad h > h_0.
\end{equation}

By $\Sigma_p(\mathbb{R}^n)$ is denoted another subset of $\Sigma(\mathbb{R}^n)$ defined as follows, $c \in \Sigma_p(\mathbb{R}^n)$ if and only if for some $C > 0, d > 0, k > 0$ and $h_0 = (h_{0,1}, \ldots, h_{0,n}) \in \mathbb{R}^n_+$:

\begin{equation}
(P.1)
\quad c(h + r) \leq Cc(h)(1 + \|r\|)^k, \quad h > h_0, \quad r \in \mathbb{R}^n.
\end{equation}

\begin{equation}
(P.2)
\quad c(h)(1 + \|h\|)^k \geq d, \quad h > h_0.
\end{equation}

Obviously, $\Sigma_p(\mathbb{R}^n) \subset \Sigma_c(\mathbb{R}^n)$.

We begin with some properties of the sets $\Sigma_c(\mathbb{R}^n)$ and $\Sigma_p(\mathbb{R}^n)$, we restrict our attention to $n = 1$ in order to analyze explicit representations for their elements.

The set $\Sigma_p(\mathbb{R})$.

By Remark 3 after Proposition 1.2, a function $c$ from Definition 1.1 has the form $c(h) = \exp(\alpha h)L(\exp(h))$, $h > h_0 > 0$, where $\alpha \in \mathbb{R}$ and $L$ is a slowly varying function, and the limit distribution $U(x) = C\exp(\alpha x)$. For the S-asymptotics in $\mathcal{S}'$, $\alpha$ has to be zero: Then $c(h) = L(\exp h)$, $h > h_0$.

**Proposition 1.10.** Suppose that $c \in \Sigma(\mathbb{R})$. 
a) If there exists $T \in \mathcal{S}'(\mathbb{R})$ such that $T(x+h) \sim c(h)U(x), h \in \mathbb{R}_+$ with $U \neq 0$, then $c(h) = L(\exp h)$ where $L$ is a slowly varying function, and so $c$ satisfies (P.2).

b) If $c = h^\nu L(h), h > h_1$ and $c(h) = 1, h \leq h_1$, where $\nu \in \mathbb{R}, h_1 > 0$ and $L$ is a slowly varying and monotonous function for $h > h_1$, then $c$ satisfies (P.1).

c) If $c$ is of the same form as in b), where $L$ is only slowly varying, then $c$ satisfies (P.1) but for $r \in \mathbb{R}_+$.

For the proof cf. [107].

The set $\Sigma_c(\mathbb{R})$.

**Proposition 1.11.** Suppose that $c \in \Sigma(\mathbb{R})$.

a) If there exists $T \in \mathcal{D}'(\mathbb{R})$ such that $T(x+h) \sim c(h)U(x), U \neq 0$, then (E.2) holds for $c$.

b) If $c(h) = \exp(\alpha|h|)^{\nu} L(h), h > h_1, c(h) = 1, h \leq h_1$, where $\alpha, \nu \in \mathbb{R}, h_1 > 0$ and $L$ is a monotonous slowly varying function for $h > h_1$, then (E.1) holds for $c$.

c) If $c$ is of the same form as in b), where $L$ is only a slowly varying function, then $c$ satisfies (E.1) but for $r \in \mathbb{R}_+$.

For the proof cf. [107].

We prove now that if $T \in \mathcal{K}'_1(\mathbb{R})$ ($T \in \mathcal{S}'(\mathbb{R})$) has the $S$-asymptotics in the cone $\mathbb{R}_+$ related to some $c(h) \in \sum_c(\mathbb{R}) (c(h) \in \sum_p(\mathbb{R}))$ with limit $U$ in the space $\mathcal{D}'(\mathbb{R})$, then the limit

$$\lim_{h \to \infty} T(x+h)/c(h) = U(x)$$

also exists in $\mathcal{K}'_1(\mathbb{R})$ (in $\mathcal{S}'(\mathbb{R})$). If $n > 1$, we have to assume some additional conditions which imply the same assertion for the multidimensional case.

First, we shall prove a theorem which extends Proposition 1.7 b) and which will be used later.

**Theorem 1.15.** Let $T \in \mathcal{D}'$. If for every rapidly exponentially decreasing function $r$ (for every $k > 0, r(x) \exp(k||x||) \to 0$ as $||x|| \to \infty$) the set \{r(h)T(x+h); h \in \mathbb{R}_n\} is bounded in $\mathcal{D}'$, then $T \in \mathcal{K}'_1$. 
1. $S$-asymptotics in $\mathcal{F}_g$

Proof. Let $K$ be an arbitrary compact set in $\mathbb{R}^n$. For every $\phi \in \mathcal{D}_K \subset \mathcal{D}$,

$$h \mapsto r(h)\langle T(x+h), \phi(x) \rangle, \ h \in \mathbb{R}^n,$$

is a bounded function. This implies that for some $k_1 = k_1(\phi) > 0$ and some $C = C(\phi) > 0$

$$\langle T(x+h), \phi(x) \rangle \leq C\exp(k_1\|h\|), \ h \in \mathbb{R}^n. \ (1.41)$$

If $h \in \mathbb{R}^n$ is fixed, then

$$\log^+|\langle T(x+h), \phi(x) \rangle |/ (1 + \|h\|), \ \phi \in \mathcal{D}_K,$$

a defines a continuous function on $\mathcal{D}_K$. It follows from (1.41) that \{ $\phi \mapsto \log^+|\langle T(x+h), \phi(x) \rangle |/ (1 + \|h\|); h \in \mathbb{R}^n$ \} is a bounded family of continuous functions on $\mathcal{D}_K$; from the classical theorem of Baire, it follows the existence of some $k > 0$, which does not depend on $\phi \in \mathcal{D}_K$, such that the set \{ $\langle T(x+h)\exp(-k\|h\|), \phi(x) \rangle; h \in \mathbb{R}^n$ \} is bounded for each $\phi \in \mathcal{D}_K$, and hence, for each $\phi \in \mathcal{D}$. This implies (see Theorem XXII, Chapter VI in [146]), that for a given open bounded set $\Omega \subset \mathbb{R}^n, 0 \in \Omega$, there exists a compact neighborhood of zero $K$ and $m \in \mathbb{N}_0$ such that for every $\phi \in \mathcal{D}^m_K$,

$$\{ x \mapsto (T(t+h) \ast \phi(t))(x)/\exp(k\|h\|); h \in \mathbb{R}^n \}$$

is a bounded family of continuous bounded functions on $\Omega$.

Since $(T(t+h) \ast \phi(t))(x) = (T \ast \phi)(x+h)$, setting $x = 0$, we obtain that $h \mapsto (T \ast \psi)(h)/\exp(k\|h\|), h \in \mathbb{R}^n$, is a bounded function on $\mathbb{R}^n$ for any $\psi \in \mathcal{D}^m_K$. Now, by (VI, 6; 22) in [146], we obtain

$$T = \Delta^N (\gamma E \ast T) - \psi \ast T, \ (1.42)$$

where $E$ is the fundamental solution of $\Delta^N E = \delta$ ( $\Delta$ is the Laplacian), $\gamma \in \mathcal{D}_K, \gamma \equiv 1$ in a neighborhood of 0 and $\psi \in \mathcal{D}_K$. If $N$ is sufficiently large, $\gamma E \in \mathcal{D}_K^m$. Thus, $\gamma E \ast T$ and $\psi \ast T$ are in $\mathcal{K}_1$, and this completes the proof. 

$\square$

**Theorem 1.16.** Let $T \in \mathcal{K}_1(\mathbb{R}) \ (T \in \mathcal{S}'(\mathbb{R}))$ and $c \in \sum_c(\mathbb{R}) \ (c \in \sum_c(\mathbb{R}))$, a) If the set $\{ T(x+h)/c(h); h > a \}$ is bounded in $\mathcal{D}'(\mathbb{R})$, then this set is bounded in $\mathcal{K}_1(\mathbb{R})$ (in $\mathcal{S}'(\mathbb{R})$) as well.

b) If there exists the limit

$$\lim_{h \to \infty} (T(x+h)/c(h), \varphi(x)) = (S, \varphi), \varphi \in \mathcal{D}(\mathbb{R}),$$

then this limit exists for every $\varphi \in \mathcal{K}_1(\mathbb{R})$ (for every $\varphi \in \mathcal{S}(\mathbb{R})$). In particular, $S \in \mathcal{K}_1(\mathbb{R})$ $\{ S \in \mathcal{S}'(\mathbb{R}) \}$. 

Proof. We prove the theorem for $T \in K'_1(\mathbb{R})$, because for $T \in S'(\mathbb{R})$ it can be done in a similar way. (We have to replace $\exp(\|\cdot\|)$ by $(1 + \|\cdot\|)^k$).

a) Using the last part of the proof of Theorem 1.15, we obtain that for some $m_1 \in \mathbb{N}_0$ and some compact neighborhood of zero, $K_1, h \mapsto (T \ast \psi)(h)/c(h), h > 0$ is a bounded function for every $\psi \in D_{m_1}^1(\mathbb{R})$. Since $T \in K'_1(\mathbb{R})$, it holds that for some $k > 0$, $m_2 \in \mathbb{N}_0$ and some compact neighborhood of zero $K_2, h \mapsto (T \ast \psi)(h)/\exp(k|h|), h \in \mathbb{R}$, is a bounded function for every $\psi \in D_{m_2}^2(\mathbb{R})$. Thus, by taking $N$ in (1.42) sufficiently large ($\Delta = d^2/dx^2$) and $K = K_1 \cap K_2$, we obtain that for some $m \in \mathbb{N}_0$

$$T = \sum_{i=0}^{m} F_i^{(i)},$$

where $F_i, i = 0, \ldots, m$, are continuous functions on $\mathbb{R}$ such that for some $M_1 > 0, M_2 > 0$ and $k > 0$

$$\sup\{|F_i(x)/c(x)|; x > 0, i = 0, \ldots, m\} \leq M_1,$$  \hspace{1cm} (1.43)

$$\sup\{|F_i(x)\exp(k|x|); x \in \mathbb{R}, i = 0, \ldots, m\} \leq M_2.$$

Let $\phi \in K_1(\mathbb{R})$ and $h > h_0$ be fixed. We put

$$I_i(h, \phi) \equiv I_i = \int_{-\infty}^{\infty} (|\phi(i)(x)||F_i(x + h)|/c(h))dx$$

$$= \left( \int_{-\infty}^{-h} + \int_{-h}^{\infty} \right) (|\phi(i)(x)||F_i(x + h)|/c(h))dx$$

$$= I_i(-\infty, -h) + I_i(-h, \infty), i = 0, \ldots, m, h > h_0.$$

If $x \in (-\infty, -h)$, then $|x + h| = |x| - h$ and by (E.2), we obtain

$$I_i(-\infty, -h) \leq d^{-1} \int_{-\infty}^{-h} |\phi(i)(x)| \exp(2k|x|)|F_i(x + h)|/\exp(k|x + h|)dx$$

$$\leq Md^{-1} \int_{-\infty}^{\infty} |\phi(i)(x)| \exp(k|x|)dx.$$

From the definition of the space $K_1(\mathbb{R})$ it follows that the last integral is finite.
Because of (E.1) and (1.43), we obtain that for some $k_1 > 0$,

$$I_i(-h, \infty) \leq \int_{-h}^{\infty} |\phi^{(i)}(x)| F_i(x + h)/c(x + h)|\exp(k_1|x|)dx$$

$$\leq M_1 \int_{-\infty}^{\infty} |\phi^{(i)}(x)| \exp(k_1|x|)dx < \infty.$$  

Since

$$\langle T(x + h)/c(h), \phi(x) \rangle \leq \sum_{i=0}^{m} I_i(h, \phi),$$

from the preceding inequalities, we obtain that for some $A > 0$ which does not depend on $h > h_0$ and for $\phi \in K_1$,

$$\langle T(x + h)/c(h), \phi(x) \rangle \leq A, \ \phi \in K_1, \ h > h_0.$$  

Thus the proof of a) is complete.

b) If the limit given in b) exists, then for an $a \in \mathbb{R}$, the set $\{T(x + h)/c(h); h \geq a\}$ is bounded in $\mathcal{K}'(\mathbb{R})$ (in $\mathcal{S}'(\mathbb{R})$). Since $\mathcal{D}(\mathbb{R})$ is dense in $\mathcal{K}_1(\mathbb{R})$ (in $\mathcal{S}(\mathbb{R})$), it is now enough to use the assertion of a) and the Banach–Steinhaus theorem. □

If we assume more on a distribution $T$, then we can assume less on the function $c$.

**Theorem 1.17.** Let $T \in \mathcal{D}'(\mathbb{R})$ and $\text{supp} T \subset [0, \infty)$. Suppose that $c \in \Sigma(\mathbb{R})$ and satisfies (E.2) and (E.1) with $r \in \mathbb{R}_+$ (satisfies (P.2) and (P.1) with $r \in \mathbb{R}_+$).

a) If the set $\{T(x + h)/c(h); h > a\}$ is bounded in $\mathcal{D}'(\mathbb{R})$, then this set is bounded in $\mathcal{K}'_1(\mathbb{R})$ (in $\mathcal{S}'(\mathbb{R})$), as well.

b) If there exists the limit

$$\lim_{h \to \infty} \langle T(x + h)/c(h), \varphi(x) \rangle = \langle S, \varphi \rangle, \ \varphi \in \mathcal{D}(\mathbb{R}),$$

then this limit exists for every $\varphi \in \mathcal{K}_1(\mathbb{R})$ (for every $\varphi \in \mathcal{S}(\mathbb{R})$). In particular, $S \in \mathcal{K}'_1(\mathbb{R})$ ($S \in \mathcal{S}'(\mathbb{R})$).

For the proof see [106].
We give other versions of Theorem 1.17 because we want to emphasize that the space $S'(\mathbb{R})$ is “natural” for the distributions having the S-asymptotics related to $c(h) = h^\nu L(h), \ h > 0, \ \nu \in \mathbb{R}$, while the space $\mathcal{K}'(\mathbb{R})$ is “natural” for those distributions having the S-asymptotics related to $c(h) = \exp(ah)L(\exp(h)), \ h > 0, \ a \in \mathbb{R}$, where $L$ is a slowly varying function.

**Theorem 1.18.** Let $T \in \mathcal{D}'(\mathbb{R}), supp \ T \subset (-\omega, \infty), \ \omega > 0$ and $T(x + h) \sim e^{\alpha x}L(e^h) \cdot Ce^{\alpha x}, \ h \in \mathbb{R}_+$ in $\mathcal{D}'(\mathbb{R})$ (and $T(x + h) \sim h^\nu L(h) \cdot C, \ h \in \mathbb{R}_+$ in $\mathcal{D}'(\mathbb{R})$), then $T \in \mathcal{K}'(\mathbb{R})$ ($T \in S'(\mathbb{R})$) and $T$ has the S-asymptotics in $\mathcal{K}'(\mathbb{R})$ (in $S'(\mathbb{R})$) related to the same $c$ and with the same limit.

**Proof.** By Theorem 1.9 and Lemma 1.2 there exist functions $F_i, \ i = 0, \ldots, m$, continuous on $(-\omega, \infty)$, such that

$$T = \sum_{i=0}^{m} D^i F_i \ on \ (-\omega, \infty),$$

where

$$|F_i(x + h)/c(h)| \leq M_i, \ h \in \mathbb{R}_+, \ x \in (-\omega, \omega), \ i = 0, \ldots, m,$$

$$supp F_i \subset (-\omega - \delta, \infty), \ i = 0, \ldots, m, \ \delta > 0.$$

For a slowly varying function $L$ there exists a slowly varying function $L^* \in C_{(c, \infty)}(\mathbb{R}), \alpha > 0$, such that $L^*(x)/L(x) \to 1, x \to \infty$. We refer to the Remark after Proposition 1.2 for the construction of $L^*$. Therefore, we can suppose, without any restriction, that $L$, which appeared in $c$, is in $C_{(c, \infty)}(\mathbb{R}), \alpha > 0$. Let us denote by $M_i(h) = F_i(h)/c(h), h \in \mathbb{R}; M_i$ is a continuous and bounded function on $\mathbb{R}$ and $F_i = cM_i$. This implies that $T \in \mathcal{K}'$. The next step is to prove that for $h_0 \in \mathbb{R}$ the set $\{T(x + h)/c(h); h \geq h_0\}$ is weakly bounded in $\mathcal{K}'$ which is equivalent to the strong boundedness in $\mathcal{K}'$.

We need the following inequality:

$$c(x + h)/c(h) = e^{\alpha x}L(e^x e^h)/L(e^h) \leq A e^{\alpha x + \delta|x|}, x, h \in \mathbb{R},$$

where $\delta > 0$ (see [9], p. 25). Let $\varphi \in \mathcal{K}'$, then

$$|\langle T(x + h)/c(h), \varphi(x) \rangle| \leq \sum_{i=0}^{m} \int_{\mathbb{R}} |F_i(x + h)/c(h)||\varphi^{(i)}(x)|dx.$$
1. S-asymptotics in $F'_g$

\[ \leq \sum_{i=0}^{m} \int_{\mathbb{R}} |M_i(x + h) c(x + h) / c(h)| |\varphi^{(i)}(x)| dx \]

\[ \leq \sum_{i=0}^{m} \int_{\mathbb{R}} e^{ax + \delta |x|} |\varphi^{(i)}(x)| dx. \]

This proves that \{\{ T(x + h) / c(h); h \geq h_0 \}\} is weakly bounded in $K'_1$. Since $D$ is dense in $K'_1$, the Banach–Steinhaus theorem implies the S-asymptotics in $K'_1$ related to $c$. □

In order to extend Theorem 1.17 to the multidimensional case, we have to introduce the following notation.

We denote by $\Lambda$ the set of all $n-th$ class variations of elements $\{-1, 1\}$. If $(a_1, \ldots, a_n) \in \Lambda$, then we put

\[ \Gamma(a_1, \ldots, a_n) = \{ h \in \mathbb{R}^n; \sum_{i=1}^{n} \text{sgn}(a_i \cdot h_i) = n \}. \]

(This means if $a_i = 1, (a_i = -1)$, then $h_i > 0 (h_i < 0)$). For example $\Gamma(1, \ldots, 1) = \mathbb{R}_{+}^n$ and $\Gamma(-1, \ldots, -1) = \mathbb{R}_{-}^n$. Let

\[ c(h) = c_1(h_1) \ldots c_n(h_n), \quad h_i \in \mathbb{R}, \quad i = 1, \ldots, n, \quad (1.44) \]

where

\[ c_i \in \Sigma_c(\mathbb{R}) \quad (c_i \in \Sigma_p(\mathbb{R})), \quad i = 1, \ldots, n. \]

Obviously, $c \in \Sigma_c(\mathbb{R}^n) (c \in \Sigma_p(\mathbb{R}^n))$. Let $(a_1, \ldots, a_n) \in \Lambda$ be given. We denote by $j_i, i = 1, \ldots, r$ the components of $(a_1, \ldots, a_n)$ which are equal to 1, and those which are equal to -1 by $s_i, i = 1, \ldots, m (r + m = n)$.

Let $h \in \Gamma(a_1, \ldots, a_n)$ and $k > 0$. We put

\[ c^k_{(a_1, \ldots, a_n)}(h) = c_{j_1}(h_{j_1}) \ldots c_{j_r}(h_{j_r}) \cdot \exp(k(|h_{s_1}| + \cdots + |h_{s_m}|)) \]

Theorem 1.19. Let $T \in K'_1 \ (T \in S')$ and $c$ be of the form (1.44).

a) If there exists $k > 0$ such that for every $(a_1, \ldots, a_n) \in \Lambda$

\[ \{ T(x + h) / c^k_{(a_1, \ldots, a_n)}(h); \quad h \in \Gamma(a_1, \ldots, a_n) \} \]

is bounded in $D'$, then \{ \{ T(x + h) / c(h); \quad h > 0 \}\} is bounded in $K'_1$ (in $S'$).

b) If

\[ \lim_{h \to \infty} \langle T(x + h) / c(h), \phi(x) \rangle = \langle S(x), \phi(x) \rangle, \phi \in D, \]
and if for some \( k > 0 \) and every \( (a_1, \ldots, a_n) \in \Lambda \setminus \{(1, \ldots, 1), (-1, \ldots, -1)\} \) the sets
\[
\{ T(x+h)/c^k(a_1, \ldots, a_n)(h); \ h \in \Gamma(a_1, \ldots, a_n) \},
\]
are bounded in \( \mathcal{D}' \), then \( T(x+h)/c(h) \) converges to \( S \) in \( K'_1 \) (in \( S' \)), as \( h \to \infty \). In particular, \( S \in K'_1 \) (\( S \in S' \)).

For the proof see [106].

Instead of analyzing the two special subspaces \( S \) and \( K_1 \) of \( \mathcal{D} \) we can consider a general subspace \( A \) of \( \mathcal{D} \).

Let \( \Gamma \) be a convex cone. We denote \( \Sigma_q(\Gamma) \) a subset of \( \Sigma(\Gamma) \) such that \( c \in \Sigma_q(\Gamma) \) if and only if there exist \( C > 0 \) and a positive locally integrable function \( p \) such that
\[
c(h+x) \leq Cc(h)p(x), \quad h, x \in \Gamma \setminus B(0, r).
\]
(1.45)

In the sequel, put \( G = \{ x \in \mathbb{R}^n \setminus (\Gamma \cup B(0, r)) \} \). We denote by \( A \) a barrelled vector space of smooth functions such that \( \mathcal{D} \) is dense in \( A \) with its topology finer than the topology induced by \( A \); \( A' \) is the dual space of \( A \), \( A' \subset \mathcal{D}' \).

We will suppose that the elements \( \phi \) of \( A \) satisfy the following condition: for every \( y \in B(0, r) \), \( p(x)\phi(x+y) \in L^1 \).

**Theorem 1.20.** Suppose that \( T \in A' \) and \( c \in \Sigma_q(\Gamma) \).

\( a) \) If the sets:
\[
Q_1 = \{ T(x+h)/c(h); \ h \in \Gamma \}
\]
\[
Q_2 = \{ T(x+k+h)/(c(h)p(k)); \ h \in \Gamma, k \in G \}
\]
are weakly bounded in \( \mathcal{D}' \), then the set \( Q_1 \) is weakly bounded in \( A' \) as well.

\( b) \) If \( T \) has \( S \)-asymptotics in \( \mathcal{D}' \) related to \( c(h) \) with the limit \( U \) and if the set \( Q_2 \) is weakly bounded in \( \mathcal{D}' \), then \( T \) has the \( S \)-asymptotics in \( A' \) related to \( c(h) \) and with the limit \( U \).

We remark that the well-known basic spaces as \( K_1, S, \mathcal{D}_{L^p}(1 \leq p < \infty) \) and \( B \) satisfy our conditions assumed in Theorem 1.20 for the space \( A \). The space \( B = \mathcal{D}_{L^\infty} \) is an example of one which does not satisfy them, due to the fact that \( \mathcal{D} \) is not dense in \( B \).
1. S-asymptotics in $\mathcal{F}_g'$

If a distribution $f$ has the S-asymptotic expansion in $\mathcal{D}'$, with adequate additional conditions, it can have the S-asymptotic expansion in a subspace of $\mathcal{D}'$. The following theorem gives such conditions.

**Theorem 1.21.** Suppose that $f \in \mathcal{K}'_1(\mathbb{R}) \ (f \in \mathcal{S}'(\mathbb{R}))$ and $\{u_i\} \subset \mathcal{K}'_1(\mathbb{R})$ ($\{u_i\} \subset \mathcal{S}'(\mathbb{R})$). If

$$f(x + h) \sim \sum_{i=1}^{\infty} u_i(x + h)\{c_i(h)\}, \quad h \to \infty \quad \text{in} \quad \mathcal{D}'(\mathbb{R})$$

and $c_i(h) = \exp(\alpha_i h)L_i(\exp h)$ (resp. $c_i = h^{\nu_i}L_i(h)$, $h > h_0$), where $L_i$, $i \in \mathbb{N}$, are monotonous slowly varying functions, then

$$f(x + h) \sim \sum_{i=1}^{\infty} u_i(x + h)\{c_i(h)\}, \quad h \to \infty, \quad \text{in} \quad \mathcal{K}'_1(\mathbb{R}) \ (\text{in} \ \mathcal{S}'(\mathbb{R})),$$

as well.

*Proof.* The proof is based on Proposition 1.11 and Theorem 1.18, we leave the details to the reader. \(\square\)

The next two propositions give the relation between the asymptotic behavior of functions and the S-asymptotics of distributions defined by these functions.

**Proposition 1.12.** Let $f \in L^1_{\text{loc}}(\mathbb{R})$ be such that $f$ defines a regular distribution in $\mathcal{K}'_1(\mathbb{R})$, i.e., $f \phi \in L^1(\mathbb{R})$ for every $\phi \in \mathcal{K}_1(\mathbb{R})$. Further, assume

$$f(x) \sim L(\exp x)\exp(\alpha x) \quad \text{as} \quad x \to \infty,$$

where $L$ is a slowly varying monotonous function defined on $(a, \infty)$. If $c(h) = L(\exp h)\exp(\alpha h), \ h > A$, and $g(x) = \exp(\alpha x), x \in \mathbb{R}$, then

$$f(x + h) \sim c(h)g(x), \ h \in \mathbb{R}^+ \ \text{in} \ \mathcal{K}'_1.$$

*Proof.* Let $\phi \in \mathcal{D}$ and supp $\phi \subset [a, b]$. Since $(L(\lambda h)/L(h)) \to 1, \ h \to \infty,$ uniformly on any compact interval contained in $(0, \infty)$, we obtain

$$\int_{-\infty}^{\infty} \frac{f(x + h)}{\exp(\alpha h)L(\exp h)}\phi(x)dx = \int_{a}^{b} \frac{f(x + h)}{\exp(\alpha(x + h))L(\exp(x + h))}$$

$$\cdot \exp(\alpha x)\frac{L(\exp x \cdot \exp h)}{L(\exp h)}\phi(x)dx \to \int_{a}^{b} \exp(\alpha x)\phi(x)dx, \ h \to \infty.$$
The assertion follows now from Proposition 1.11 and Theorem 1.16. □

**Proposition 1.13.** Suppose that the functions \( f \) and \( u_i, i \in \mathbb{N} \), are locally integrable and define regular distributions belonging to \( K'_1(\mathbb{R}) \), i.e., \( f \varphi \) and \( u_i \varphi \) are in \( L^1(\mathbb{R}) \) for every \( \varphi \in K_1(\mathbb{R}) \). Let \( c_i(h) = \exp(\alpha_i h) L_i(\exp h) \), where \( L_i \) are monotonous slowly varying functions and \( \alpha_i \in \mathbb{R}, i \in \mathbb{N} \). If
\[
f(x) \sim \sum_{i=1}^{\infty} U_i(x) \{c_i(x)\}, \quad x \to \infty,
\]
then
\[
f(x + h) \sim \sum_{i=1}^{\infty} u_i(x + h) \{c_i(h)\}, \quad h \to \infty.
\]

*Proof.* The proof is similar to that of Proposition 1.13. □

We shall prove a structural theorem for a distribution \( T \in B' \) having the S-asymptotics in a cone \( \Gamma \) with the nonempty interior.

**Theorem 1.22.** Suppose \( T_0 \in B' \) and \( T_0(x + h) \sim 1 \cdot U(x), \ h \in \Gamma \) in \( D' \), then

a) \( U = C \).

b) \( T_0 = \sum_{i=0}^{2} \Delta^i F_i \), where \( F_i \) are continuous functions belonging to \( L^\infty \);

c) For every \( 0 \leq i \leq 2 \) functions \( F_i(x + h) \) of part b) converge uniformly to a constant when \( x \) belongs to a compact set \( K \) and \( h \in \Gamma, \|h\| \to \infty \).

d) \( T_0 \) has the S-asymptotics in \( B' \), related to \( c = 1 \) and with the limit \( U = C \) in the cone \( \Gamma \).

*Proof.* a) \( U \) has to be a constant because \( c = 1 \) (Proposition 1.2).

b) From the fact that \( T_0 \in B' \) it follows that \( (T_0 * \hat{\zeta})(x + \cdot), \zeta(x) \in L^\infty \) for every \( \zeta \in D \) (see [146], VI, §8) and the set of distributions \( Q = \{T_h \equiv T_0(x + h); \ h \in \mathbb{R}^n \} \) is weakly bounded and bounded in \( D' \).

We will construct another bounded set of distributions. Denote \( S = \{\psi \in D; \|\psi\|_{L^1} \leq 1 \} \). We have seen that for a fixed \( \zeta \in D \), \( (T_0 * \zeta) \in L^\infty \).

Now, for every \( \psi \in S \):
\[
|\langle T_0 * \hat{\psi}, \zeta \rangle| = |\langle T_0 * \hat{\zeta}, \psi \rangle| = \int_{\mathbb{R}} (T_0 * \hat{\zeta})(t)\psi(t)dt \leq \|T_0 * \zeta\|_{L^\infty} \|\psi\|_{L^1}.
\]
1. S-asymptotics in $F_r$

Hence, the set of regular distributions, defined by the set of continuous functions $\{U_\psi \equiv T_0 * \hat{\psi}; \psi \in S\}$ is weakly bounded and therefore bounded in $\mathcal{D}'$.

A set $W' \in \mathcal{D}'$ is bounded if and only if for every $\alpha \in \mathcal{D}$ the set of functions $\{T * \alpha; T \in W'\}$ is bounded on every compact set $M$ belonging to $\mathbb{R}^n$ (see [146], VI, §7). Hence $\{T * \alpha; T \in W'\}$ defines a bounded set of regular distributions. In such a way $\{T_h * \zeta; T_h \in Q\}$ and $\{U_\psi * \zeta; U_\psi \in H\}$ are bounded sets of regular distributions. Now, for these two sets we can repeat twice a part of the proof of Theorem XXII, Chapter VI in [146].

We denote by $\Omega$ an open neighborhood of zero in $\mathbb{R}^n$ which is relatively compact in $\mathbb{R}^n$, namely, $\Omega = K$ is a compact set. Then, by the mentioned part of the proof, there exist $m_1 \geq 0$ and $m_2 \geq 0$, such that the mappings $(\alpha, \beta) \rightarrow U_\psi * (\alpha * \beta)$ or $(\alpha, \beta) \rightarrow T_h * (\alpha * \beta)$, $\alpha, \beta \in \mathbb{R}^n$, are equicontinuous and map $\mathcal{D}_i^{m_1} \times \mathcal{D}_i^{m_2}$ and $\mathcal{D}_i^{m_1} \times \mathcal{D}_i^{m_2}$ into $L_\infty$ respectively; $B$ is the ball $B(0, r)$, where $r$ is a positive number. Hence, for every $x \in B$ and $h \in \mathbb{R}^n$ the function $(T_h * \alpha * \beta)(x) = (T_0 * \alpha * \beta)(x + h)$ is continuous.

Let $Z(0, \rho)$ be a ball in $L_\infty$. Then there exists a neighborhood $V_1(m_1, \varepsilon_1, K_1)$ in $\mathcal{D}_i^{m_1}$, such that $U_\psi * (\alpha * \beta) \in Z(0, \rho)$ for $\alpha, \beta \in V_1(m_1, \varepsilon_1, K_1)$, $U_\psi \in H$, and a neighborhood $V_2(m_2, \varepsilon_2, K_2)$ in $\mathcal{D}_i^{m_2}$, such that $T_h * (\alpha * \beta) \in Z(0, \rho)$ for $\alpha, \beta \in V_2(m_2, \varepsilon_2, K_2)$, $T_h \in Q$. Let $K_0 = K_1 \cap K_2$, $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$ and $m = \max(m_1, m_2)$. We shall use (VI, 6; 23) in [146].

\[
T_0 = \Delta^{2k} * (\gamma E * \gamma E * T_0) - 2\Delta^k * (\gamma E * \xi * T_0) + (\xi * \xi * T_0),
\]

where $E$ is a solution of the iterated Laplace equation; $\Delta^k E = \delta$; $\gamma, \xi \in \mathcal{D}_0$, sup $\gamma$ and sup $\xi$ belonging to $K_0 = K_1 \cap K_2$. We have only to choose the number $k$ large enough so that $\gamma E \in \mathcal{D}_i^{m_0}$. Now, we can take: $F_2 = \gamma E * \gamma E * T_0$; $F_1 = \gamma E * \xi * T_0$ and $F_0 = \xi * \xi * T_0$. All of these functions are of the form: $F_i = T_0 * \alpha_i * \beta_i$; $\alpha_i, \beta_i \in V(m, \varepsilon_0, K_0)$, $\varepsilon_0 > 0$.

We have to prove that $F_i, \ i = 0, 1, 2$, have the properties given in Theorem 1.22. For $\alpha_i, \beta_i \subset V(m, \varepsilon_0, K_0)$ and $\psi \in S$

\[
||(T_0 * \alpha_i * \beta_i, \psi)|| = ||(T_0 * \hat{\psi}) * (\alpha_i * \beta_i)||[0] \leq \rho(\varepsilon_0/\varepsilon_0)^2 \equiv M.
\]

Now let $\mu \not= 0$ be any element in $L^1$. Then $\mu/||\mu||_{L^1} \in S$ and $||(T_0 * (\alpha_i * \beta_i, \mu)|| \leq M||\mu||_{L^1}$ which proves that $T_0 * (\alpha_i * \beta_i), \ i = 0, 1, 2$, belong to $L_\infty$. Since $F_i = T_0 * (\alpha_i * \beta_i), \ \alpha_i, \beta_i \in V(m, \varepsilon_0, K_0), \ F_i, \ i = 0, 1, 2$, are continuous and belong to $L_\infty$. 


c) We shall continue the investigations of the properties of $F_i$. By the properties of the convolution, we have

$$F_i(x + h) = F_i * \tau_{-h} = T_0 * (\alpha_i * \beta_i) * \tau_{-h} = T_h * (\alpha_i * \beta_i),$$

where $\tau_{-h}$ is the translation operator.

We have proved that the mappings $(\alpha, \beta) \mapsto T_h * \alpha * \beta, T_h \in Q$, are equicontinuous and map $D^m \Omega \times D^m \Omega$ into $L^\infty B$. $D$ is a dense subset of $D^{\infty}$, $m \geq 0$.

We can construct a subset of $D^K, \text{cl} \Omega = K$, which is dense in $D^m \Omega$. Since $T_h * (\zeta * \psi) \to C * \zeta * \psi$ for $\zeta * \psi \in D_\Omega \times D_\Omega$, then $T_h * \alpha_i * \beta_i$ converges to $C * \alpha_i * \beta_i$, as well (see [146], VI, §7), when $\|h\| \to \infty$, $h \in \Gamma, i = 0, 1, 2$.

d) It remains to prove the last part of Theorem 1.22. For $\mu \in D_{L^1}$ and $T \in B$, we have:

$$|\langle T_0(x + h), \mu(x) \rangle| \leq 2 \sum_{i=0}^{\infty} \int_{\mathbb{R}^n} |F_i(x + h) \Delta^{ik} \mu(x)| dx \leq 2 \sum_{i=0}^{\infty} M_i \int_{\mathbb{R}^n} |\Delta^{ik} \mu(x)| dx,$$

where $M_i = \sup |F_i(x)|, x \in \mathbb{R}^n$. Hence the set $\{T_0(x + h), h \in \mathbb{R}^n\}$, is weakly bounded in $B'$. Since $D$ is dense in $D_{L^1}$, by the Banach–Steinhaus theorem the limit:

$$\lim_{h \in \Gamma, \|h\| \to \infty} (T_0(x + h), \mu(x)), \mu \in D_{L^1},$$

exists, as well, and it is equal to $\langle C, \mu \rangle$. □

### 1.12 Generalized S-asymptotics

Definition 1.1 corresponds to some subsets of distributions of a limited growth (see Theorem 1.16). This is the reason for introducing a generalization of the S-asymptotics ([119]).

**Definition 1.6.** Suppose that $c$ is a function on $\mathbb{R}^n$ such that $c^{-1} \in M(\cdot)$ (see 0.6). Then, it is said that $T \in \mathcal{F}^\gamma_c$ has generalized S-asymptotics in the cone $\Gamma$, related to $c$, if there exists

$$w. \lim_{\|h\| \to \infty, h \in \Gamma} T(x + h) / c(x + h) = 1 \text{ in } \mathcal{F}^\gamma_c.$$

We write in short $T(x) \sim^c e(x), \|x\| \to \infty, x \in \Gamma$.

If we compare Definition 1.1 and Definition 1.6, we obtain.
Proposition 1.14. Let $e$ be a function on $\mathbb{R}^n$ such that $e^{-1} \in M(\cdot)$. Then, $T \in F_g' \Gamma$ has generalized S-asymptotics in $\Gamma$, related to $e$ if and only if $T/e$ has the S-asymptotics in $\Gamma$ related to $c = 1$ and with limit $U = 1$.

In such a way, dividing $T \in F_g'$ by $e$, we may apply our results on the S-asymptotics to this new context. So, all the assertions on the S-asymptotics can be transferred for the generalized S-asymptotics by simply using Proposition 1.14. For this reason, we underline only some of the key properties of generalized S-asymptotics.

The generalized S-asymptotics is a local property, as well (see Theorem 1.4). From Theorems 1.2,d) and 1.3,4) it follows:

Proposition 1.15. a) Let $S \in E'$ and $T \in D'$. If $T(x) \同等\ e(x), \|x\| \to \infty, x \in \Gamma$, then $(S \ast (T/e))(c + h) \同等\ 1 \cdot (S \ast 1)(x), h \in \Gamma$.

As consequence of $T(x) \同等\ e(x), \|x\| \to \infty, x \in \Gamma$, it follows that for any partial derivative $D_x$,

$$(D_x(T/e))(x + h) \同等\ 1 \cdot 0, \|h\| \to \infty, h \in \Gamma.$$ 

In order to compare the existence of the generalized S-asymptotics with that of the S-asymptotics of a $T \in F_g'$, we need the following notation and lemma.

Let $\Gamma$ be a cone with the vertex at zero. We employ the notation $pr \Gamma$ for the intersection of $\Gamma$ and the unit sphere in $\mathbb{R}^n$.

Lemma 1.4. Let $T \in F_g'$ and let $c \in \Sigma(\Gamma)$ where $\Gamma$ is a convex cone with the nonempty interior, ( $\emptyset \not= \Gamma \not= \emptyset$). Let $\Gamma'$ be a closed cone, $\Gamma' \subset \text{int} \Gamma$. Suppose that $T(x + h) \同等\ c(h) \cdot U(x), h \in \Gamma$ with $U \not= 0$. Then, there exist $e \in C^\infty$, positive on $\mathbb{R}^n$, and $a \in \mathbb{R}^n$ such that

$$\lim_{h \in \Gamma, \|h\| \to \infty} e(h)/e(x + h) = \exp(-(a \cdot x)) \text{ in } E.$$ \hspace{1cm} (1.46)

Proof. We know (see Proposition 1.2) that $U(x) = C \exp(a \cdot x), a \in \mathbb{R}^n, C \not= 0$. Let $\phi_0 \in F$ such that $(U, \phi_0) \not= 0$. We introduce functions $e_1, e_2, e_3, e_4$ and $e$ in the following way: $e_1$ is a smooth function $e_1(y) = (T(x + y), \phi_0(x))/\langle U, \phi_0 \rangle, y \in \mathbb{R}^n$.

Since

$$\lim_{h \in \Gamma, \|h\| \to \infty} (T(x + h)/e(h), \phi_0(x)) = \langle U, \phi_0 \rangle,$$
there exists $\beta_0 > 0$ such that $e_1(y) > 0, y \in \{z \in \Gamma', \|z\| \geq \beta_0\}$.

Functions $e_2, e_3$ and $e_4$ are defined as follows: $e_2(x) = \max\{e_1(x), 0\}$, $x \in \mathbb{R}^n$, $e_3$ is the characteristic function of the set $C_{\mathbb{R}^n}\{x \in \Gamma', \|x\| \geq \beta_0\}$, $e_4 = e_2 + e_3$. Thus $e_4$ is positive and locally integrable.

Let $w$ be a non-negative function belonging to $C^\infty$ such that $w(x) = 1$ for $x \in B(0, 1)$, $w(x) = 0$ for $\|x\| > 2$, and

$$\int_{\mathbb{R}^n} \exp(-a \cdot x))w(x)dx = 1.$$

Now, we can construct the sought function $e \in C^\infty : e = (e_4 \ast w)(x)$, $x \in \mathbb{R}^n$. The function $e(x)$ is positive because of

$$e(x) = \int_{\mathbb{R}^n} e_4(x-t)w(t)dt \geq \int_{B(0,1)} e_4(x-t)dt > 0, x \in \mathbb{R}^n.$$

It only remains to prove (1.45). The number $m := d(pr \Gamma', C_{\mathbb{R}^n}, \text{int } \Gamma)$ is positive because $pr \Gamma'$ is a compact set in $\mathbb{R}^n$ and $C_{\mathbb{R}^n}$ in $\text{int } \Gamma$ is a closed one. Every ball $B(b, m')$, $0 < m' < \min(m, 1)$, $b \in pr \Gamma'$ is contained in $\text{int } \Gamma$ and for every $\lambda > 0$ the ball $B(\lambda b, \lambda m')$, $0 < m' < \min(m, 1)$, is contained in $\text{int } \Gamma$, as well.

For every compact set $K \subset \mathbb{R}^n, K \subset B(0, \rho)$ and every $\beta_0 \in \mathbb{R}_+$ there exists $\beta > 0$ such that $x + h \in \{y \in \Gamma'; \|y\| \geq \beta_0\}$ for $x \in K$ and $h \in \{y \in \Gamma', \|y\| > \beta\}$. Since $h = \lambda h$, $h$ can be written as $h = \lambda h, b \in pr \Gamma', \lambda > 0$. Thus $B(\lambda b, \lambda m') \subset \Gamma$ for every $\lambda > 0$. If $\lambda > \rho/m'$, then $x + h \in B(\lambda b, \lambda m')$ for $x \in K \subset B(0, \rho)$ and $h = \lambda b$ because $\|x + h - \lambda b\| = \|x\| \leq \rho < \lambda m'$. Now,

$$\|x + h\| \geq \|x\| - \|h\| \geq \lambda(1 - m').$$

Then, we can take $\beta = \max\{\rho/m', \beta_0/(1 - m')\}$.

For a compact set $K_0 \subset \mathbb{R}^n$ the set $K = K_0 \setminus B(0, 2)$ is also a compact set and belongs to a ball $B(0, \rho)$. Let us suppose that we have found $\beta$ which corresponds to $K$ and $\beta_0$ as above. Then, by the definition of $e_4$ we have $e_4(x - t + h) = e_1(x - t + h)$ for $x - t \in K; h \in \Gamma', \|h\| \geq \beta$. Hence

$$\lim_{h \in \Gamma', \|h\| \to \infty} e_4(x - t + h)/(c(h)) = \lim_{h \in \Gamma', \|h\| \to \infty} \left\langle \frac{T(y + x - t + h)}{c(h)(U, \phi_0)}, \phi_0(y) \right\rangle = \exp(a \cdot (x - t))$$
and this limit is uniform for \( x - t \in K \). Now, for \( x \in K_0 \) the following limit
\[
\lim_{h \in \Gamma', \|h\| \to \infty} \frac{e(x + h)}{c(h)} = \lim_{h \in \Gamma', \|h\| \to \infty} \left( \frac{e(t + h)}{c(h)} \ast w(t) \right)(x)
\]
\[
= \left( \exp(a \cdot t) \ast w(t) \right)(x) = \exp(a \cdot x)
\]
is uniform, as well. \( \square \)

**Proposition 1.16.** Let \( \Gamma \) be a convex cone with nonempty interior, \( \int \Gamma \neq 0 \), and let \( \Gamma' \) be a closed cone such that \( \Gamma' \cap \partial B(0, 1) \subset \text{int} \Gamma \). Let \( c \in \Sigma(\Gamma) \) and \( T \in F' \). If for \( U(x) = C \exp(\alpha \cdot x), \ x \in \mathbb{R}, \ C \in \mathbb{R}, \ C \neq 0, \)
\[
T(x + h) \sim c(h) \cdot U(x), \ h \in \Gamma, \ U \neq 0,
\]
then there exists \( e \in C^\infty \) such that \( e(x) \neq 0, \ x \in \mathbb{R}^n, \) and \( T(x) \ \not\sim C e(x), \|x\| \to \infty, \ x \in \Gamma' \).

**Proof.** Let \( e \) be as in Lemma 1.4. It is enough to apply Proposition 1.2 and Theorem 1.2, b) to
\[
\langle T(x + h)/e(x + h), \varphi(x) \rangle = \left\langle \frac{c(h)}{e(x + h)} \cdot T(x + h), \varphi(x) \right\rangle, \varphi \in F. \quad \square
\]

The next proposition also gives a relation between the S-asymptotics and the generalized S-asymptotics.

**Proposition 1.17.** Suppose that \( T \in F', \Gamma = \{ x \in \mathbb{R}^n, x = (0, \ldots, x_k, 0, \ldots, 0) \}, T/c = D_{x_k} S \) and \( T(x) \ \not\sim a c(x), \|x\| \to \infty, \ x \in \Gamma \). Then \( S(x + h) \ \not\sim h_k \cdot 1, h \in \Gamma \).

**Proof.** By L’Hospital’s rule with Stolz’s improvement, we have (with \( h = (0, \ldots, h_k, 0, \ldots, 0) \) and \( S \ast \phi(h) = (S(x + h), \phi(x)) \))
\[
\lim_{h_k \to \infty} \langle S(x + h)/h_k, \phi(x) \rangle = \lim_{h_k \to \infty} \langle \frac{S \ast \phi(h)}{h_k} \rangle = \lim_{h_k \to \infty} \left( \frac{T}{c} \ast \phi \right)(h)
\]
\[
= \lim_{h_k \to \infty} \left\langle \frac{T(x + h)}{c(x + h)}, \phi(x) \right\rangle = \langle a, \phi(x) \rangle. \quad \square
\]

Finally, let us point out that if we take the limit
\[
\lim_{h \in \Gamma', \|h\| \to \infty} T(x + h)/(c(h)e(x + h)),
\]
in Definition 1.6., then nothing new is obtained (cf. Definition 1.6) for the following three cases: the one-dimensional case, when \( \Gamma \subset \mathbb{R}^n \) is a ray, and when \( \Gamma \) has nonempty interior.
2 Quasi-asymptotics in $\mathcal{F}'$

2.1 Definition of quasi-asymptotics at infinity over a cone

Quasi-asymptotics has been originally defined and studied for tempered distributions. The motivation for such a choice can be found in Theorem 2.3 below. The first paper dealing with the analysis of the quasi-asymptotics was written by Zavyalov [200]. Thereafter, many results concerning the theory and applications of this notion have appeared. The main features of the theory (published until the year 1986) have been collected in the monograph [192]. We start with the most general definition of quasi-asymptotics of tempered distributions ([192]).

Let $\Gamma$ be a closed convex acute solid cone (cf. 0.2) in $\mathbb{R}^n$ and let $\{U_k; k \in I \subset \mathbb{R}\}$ be a family of linear nonsingular transforms of $\mathbb{R}^n$ which leave the cone $\Gamma$ invariant (automorphisms of $\Gamma$). We assume that $J_k = \det U_k > 0$ and that $I$ has $\infty$ as a limit point. Furthermore, let $\rho(k)$ be a positive function defined on $I$. Denote by $\mathcal{S}'_\Gamma$ the space of tempered distributions with supports in the cone $\Gamma$. It is important to mention that $\mathcal{S}'_\Gamma$ is isomorphic to the space $\mathcal{S}'(\Gamma)$ (see 0.5).

Definition 2.1. Let $T \in \mathcal{S}'_\Gamma$. It is said that $T$ has the quasi-asymptotics in the cone $\Gamma$ over the family $\{U_k; k \in I\}$ related to the positive measurable function $\rho(k)$ if there exists a tempered distribution $g \neq 0$ such that
\[
\lim_{k \in I, k \to \infty} \frac{1}{\rho(k)} T(U_k x) = g(x) \quad \text{in} \quad \mathcal{S}'.
\]

We will discuss the quasi-asymptotics in a particular case, namely, if $T \in \mathcal{F}'$, $U_k \Gamma = k \Gamma$ and $I = (0, \infty)$, where $\Gamma$ is a closed, convex and acute cone in $\mathbb{R}^n$ (see [33]). For results related to Definition 2.1, see [192].

Definition 2.2. Let $T \in \mathcal{F}'_\Gamma$. It is said that $T$ has the quasi-asymptotics in the cone $\Gamma$ related to a positive measurable function $\rho$ if there exists $g \neq 0$ such that
\[
\lim_{k \to \infty} \frac{1}{\rho(k)} T(kx) = g(x) \quad \text{in} \quad \mathcal{F}'.
\]

We write in short $T(kx) \overset{q}{\sim} \rho(k) g(x)$, $k \to \infty$ in $\mathcal{F}'$; we will omit the space from the notation whenever it is clear from the context.
2. Quasi-asymptotics in $\mathcal{F}'$

The definition itself forces $\rho$ and $g$ to have very specific forms. The following proposition states precisely to which classes of functions and distributions $\rho$ and $g$ must belong.

**Proposition 2.1.** Let $T \in \mathcal{F}'_{\Gamma}$. If $T(kx) \overset{\mathcal{D}}{\rightarrow} \rho(k)g(x)$, $k \to \infty$, then:

(a) $\rho$ is a regularly varying function of the form $\rho(k) = k^\alpha L(k)$, $k \geq k_0$, (see 0.3) where $\alpha \in \mathbb{R}$ and $L$ is a slowly varying function.

(b) $g$ is a homogeneous element of $\mathcal{F}'_{\Gamma}$ with degree $\alpha$, that is, $g(kx) = k^\alpha g(x)$, $k > 0$.

**Proof.** We take $\phi \in \mathcal{F}_{\Gamma}$ such that $\langle T, \phi \rangle \neq 0$. Let $K \subset (0, \infty)$ be any compact set. Then, by the assumption, we have

$$\lim_{k \to \infty} \frac{\langle T(ktx), \rho(k) \phi(x) \rangle}{\rho(k)} = \langle g(x), \phi(x) \rangle$$

uniformly in $t$, provided that $t$ belongs to the compact set $K$. On the other hand, we have uniformly in $t \in K$

$$\lim_{k \to \infty} \frac{\langle T(ktx), \rho(k) \phi(x/t) \rangle}{t} = \langle g(x), \phi(x/t) \rangle.$$

Combining relations (2.1) and (2.2), we obtain for $t > 0$,

$$\lim_{k \to \infty} \frac{\rho(k)}{\rho(k)} = \lim_{k \to \infty} \frac{\langle T(ktx), \phi(x/t) \rangle}{\langle T(ktx), \phi(x) \rangle} = \frac{\langle g(x), \phi(x) \rangle}{\langle g(x), \phi(x) \rangle} = C(t)$$

uniformly in $t \in K$. This is just the definition of a regularly varying function (see 0.3). We thus have $C(t) = t^\alpha$, $t > 0$, for some $\alpha \in \mathbb{R}$; therefore, $\rho$ has the desired form.

In order to prove that $g$ is homogeneous of degree $\alpha$, we have to use (2.1):

$$\langle g(tx), \varphi(x) \rangle = \lim_{k \to \infty} \frac{\langle T(ktx), \rho(k) \varphi(x) \rangle}{\rho(k)}$$

$$= \lim_{k \to \infty} \frac{\rho(k)}{\rho(k)} \frac{\langle T(ktx), \varphi(x) \rangle}{\rho(k)}$$

$$= (t^\alpha g(x), \varphi(x)),$$

$\varphi \in \mathcal{F}_{\Gamma}$, $t > 0$.

**Remark.** It is easy to prove that in case $\mathcal{F}'_{\Gamma} = S_{\mathbb{R}_+}' \equiv S_{\mathbb{R}_+}'$, then $g = Cf_{\alpha+1}$, where $f_{\alpha}$ is the homogeneous tempered distribution of degree $\alpha - 1$ given in 0.4.
We emphasize that that all homogeneous distributions on the real line are explicitly known. Recall ([56]), they are of the form
\[ g(x) = C_- x^\alpha + C_+ x_+^\alpha \text{ if } \alpha \notin \mathbb{N}, \]
\[ g(x) = \gamma \delta^{(k-1)}(x) + \beta x^{-k} \text{ if } \alpha = -k \in \mathbb{N} \]
where \( C_- , C_+, \gamma, \beta \in \mathbb{C} \) and the distribution \( x^{-k} \) stands for the standard regularization of the corresponding function \([56, 63]\), i.e., \( x^{-1} = (\log |x|)' \), \( -k x^{-k-1} = (x^{-k})' \). We refer to \([40, 56, 63]\) for the explicit form of multidimensional homogeneous distributions.

An more general analysis of homogeneous generalized functions can be found in \([85]\), but only in one dimension. We comment only part of it.

Let \( W_\beta(I) \) be an abstract locally convex function space whose elements are defined on \( I = (-\infty, \infty) \) or on \( I = (0, \infty) \) and for which \( \phi(\cdot) \mapsto \phi\left(\frac{\cdot}{y}\right), \ y > 0 \), is a continuous mapping from \( W_\beta(I) \) to \( W_\beta(I) \), where \( W_\beta(I) \) is the dual space of \( W_\beta(I) \), satisfies the equation
\[ g(y) = y^\alpha g(\cdot), \ y > 0, \ \alpha \in \mathbb{R}, \]
in the sense that \( \langle g(\cdot), \phi(\cdot) \rangle = y^\alpha \langle g(\cdot), \phi(\cdot) \rangle \) for all \( \phi \in W_\beta(I) \), then \( g \) is called a homogeneous generalized function from \( W_\beta(I) \) of degree \( \alpha \).

Let us suppose that \( W_\beta(I) \) is given by the subspace of \( C^\infty(I \setminus \{0\}) \) for which all the seminorms
\[ p_{n,\beta}(\phi) = \int_I |x|^\beta |x^n \phi^{(n)}(x)| dx, \ n \in \mathbb{N}_0 \]
are finite. In particular \( p_{0,\beta} \) is a norm.

One can easily prove that for a given \( \phi \in W_\beta(I) \) the sequence \( \{n_{\eta_0}, \phi\}_{n \in \mathbb{N}} \), where \( n_{\eta_0} \) is an even positive function in \( \mathcal{D}(I) \) such that \( n_{\eta_0}(x) = 0 \) for \( x \in [0, 1/2n] \cup [n + 1, \infty) \) and \( n_{\eta_0}(x) = 1 \) for \( x \in [1/n, n] \), \( n \in \mathbb{N} \), converges to \( \phi \) in \( W_\beta(I) \) as \( n \to \infty \). This implies that the space \( \mathcal{D}(I \setminus \{0\}) \) is dense in \( W_\beta(I) \). Thus, all homogeneous generalized functions of order \( \alpha \notin -\mathbb{N} \) in \( W_\beta(I) \) are of the form
\[ A_1 x^\alpha + A_2 x_-^\alpha \text{ resp. } A_1 x_+^\alpha \text{ if } I = \mathbb{R} \text{ resp. } I = (0, \infty). \]
2. Quasi-asymptotics in $F$

Let $U_\beta(I) = \{\phi \in W_\beta(I) : \phi(x) = 0 \text{ for } x \in I \}$ be a subspace of $W_\beta(I)$ with a locally convex topology, such that the inclusion mapping $i : U_\beta(I) \to W_\beta(I)$ is continuous, as well as the following one $U_\beta(I) \to U_\beta(I) : \phi \mapsto \phi \cdot y^{-1}$, $y \in (0, \infty)$.

If $f \in W_\beta(I)$ and if it is homogeneous of order $\alpha$, then $f|_{U_\beta(I)}$, the restriction of $f$ on $U_\beta(I)$, is of the form (2.4). A natural question is then to characterize spaces for which all homogeneous generalized functions are of the form (2.4). One can show that this is the case for $U_\beta'(I)$. At a first sight, the introduction of these spaces may seem to be artificial, but it is naturally connected with the problem we are discussing.

Note that $D(I \setminus \{0\})$ need not be dense in $U_\beta(I)$. This is in fact the only interesting case. $U_\beta(I)$ may not contain $D(I \setminus \{0\})$ at all. If $U_\beta(I)$ contains $D(I \setminus \{0\})$ then clearly $U_\beta(I)$ is dense in $W_\beta(I)$. Even in that case, we could not use this fact for the proof that any homogeneous element $g$ on $U_\beta(I)$ can be extended to $W_\beta(I)$, i.e., $g \in W_\beta'(I)$, and thus it is of the form (2.4).

For $U_\beta(I)$, we assume: if $\phi \in U_\beta(I)$ and $y_0 \in (0, \infty)$, then

$$
\lim_{y \to y_0} \frac{\phi(y) - \phi(y/y_0)}{y - y_0} = \left. \frac{d}{dy} \phi(y/y_0) \right|_{y=y_0}
$$

in the sense of convergence in $U_\beta(I)$. We have the following proposition (for the proof see [85]).

**Proposition 2.2.** A generalized function $f \in U_\beta'(I)$ is homogeneous of order $\alpha \in \mathbb{R}$ if and only if for each $\phi \in U_\beta(I)$

$$
\langle f(x), (x \phi(x))' + \alpha \phi(x) \rangle = 0,
$$

i.e., if and only if $f$ fulfills the equation $xf' = \alpha f$ in $U_\beta'(I)$.

A further analysis shows that $U_\beta'(I)$ has the desired properties, for details see [85].

### 2.2 Basic properties of quasi-asymptotics over a cone

Recall, $\Gamma^* = \{y; y \cdot x \geq 0, \text{ for every } x \in \Gamma\}$ is the conjugate cone to the cone $\Gamma$ (cf. 0.2). Denote by $C = \text{int} \Gamma^*$. The characteristic function of a closed
convex solid acute cone $\Gamma$ is denoted by $\theta_{1,\Gamma}$. The function

$$K_C(z) = \int_{\Gamma} e^{iz \cdot t} dt, \quad z \in \mathbb{R}^n + iC$$

is called the Cauchy-Szegő kernel of the tube domain $\mathbb{R}^n + iC$. The closed convex acute solid cone $\Gamma$ is regular if $K_C(z)$ is a divisor of unity in the Vladimirov algebra $H(C) = \mathcal{L}(S'(\Gamma))$, where $\mathcal{L}$ denotes the Laplace transform (cf. [189], §12). In case $n = 1, 2, 3$ it is well-known (see Chapter 1, §2.7 in [192]) that all (closed convex acute solid) cones are regular. For regular cones the distribution $\theta_{\alpha,\Gamma}$ is given by

$$\int_{\Gamma} e^{iz \cdot t} \theta_{\alpha,\Gamma}(t) dt = K_C^\alpha(z), \quad z \in \mathbb{R}^n + iC, \quad \alpha \in \mathbb{R}.$$  

This distribution has many interesting properties (see [33]):

1) $\text{supp} \theta_{\alpha,\Gamma} \subset \Gamma$ and $\theta_{\alpha,\Gamma} \in S'_{\Gamma}$;
2) $\theta_{\alpha,\Gamma}(kt) = k^{\alpha(n-1)} \theta_{\alpha,\Gamma}(t), \quad t \in \Gamma, \quad k > 0$;
3) $\theta_{\alpha,\Gamma} * \theta_{\beta,\Gamma} = \theta_{\alpha+\beta,\Gamma}$;
4) For every $m \geq 0$ there exists $n_0$ such that $\theta_{\alpha,\Gamma} \in C^m(\mathbb{R}^n), \quad \alpha > n_0$;
5) $\{\text{supp} \theta_{\alpha,\Gamma}(e^{-t})\} \cap \Gamma \subset \{\|t\| < R\}$, where $R > 0$ do not depend on $\alpha$ and $e \in \text{pr} \Gamma = \Gamma \cap \{|\|x\| = 1\}$.

6) For any $p \in \mathbb{N}_0$, there exists $N_0 > 0$ such that for some $A$ and $q > 0$ and for every $n \geq N_0$, $\|\theta_{\alpha,\Gamma}(x - t)\|_{p,\Gamma} \leq A(1 + \|x\|)^q$, $x \in \Gamma$ (cf. 0.5 for $\|\|p,\Gamma\|$).

Operations of fractional derivatives (for $\alpha \leq -1$) and fractional integrals (for $\alpha > -1$) can be defined on $S'_{\Gamma}$ via

$$\theta_{\alpha,\Gamma} : T \rightarrow \theta_{\alpha,\Gamma} * T,$$

the latter defines a continuous linear operator on $S_{\Gamma}^r$. We will use the notation $T^{(\alpha)}$ for $\theta_{-\alpha,\Gamma} * T$. By the properties of the convolution, we have

$$T^{(-\alpha)}(t) = \langle T(\tau), \theta_{\alpha,\Gamma}(t - \tau) \rangle,$$

if $\alpha$ is sufficiently large.

**Definition 2.3.** Suppose that $f \in L_{loc}(\Gamma)$. It is said that $f$ has an asymptotic behavior in the cone $\Gamma$ related to the positive function $\rho(k), \quad k \in (0, \infty)$ if there exists a function $g \neq 0$ such that

$$\lim_{k \to \infty} f(kx)/\rho(k) = g(x), \quad x \in \Gamma,$$

(2.5)
2. Quasi-asymptotics in $\mathcal{F}'$

$$|f(kx)/\rho(k)| \leq h(x), \quad x \in \Gamma, \ k \geq k_0 \geq 0, \quad (2.6)$$
and $\varphi(x)h(x) \in L^1(\Gamma)$, for every $\varphi \in \mathcal{F}$.

**Proposition 2.3.** If a locally integrable function $f$ has the asymptotic behavior in the cone $\Gamma$ related to $\rho$ (Definition 2.3), and if $f$ defines a regular element of $\mathcal{F}'_\Gamma$, then $f(kx) \not\sim \rho(k)g(x), \ k \to \infty$.

**Proof.** We have
$$\lim_{k \to \infty} \langle f(kx)/\rho(k), \varphi(x) \rangle = \lim_{k \to \infty} \int_\Gamma f(kx)\varphi(x)/\rho(k)dx, \ \varphi \in \mathcal{F}.$$
The assumption over $f$ allows one to exchange the limit with the integral, by the Lebesgue theorem. □

**Proposition 2.4.** Let $\mathcal{F}$ have the property that $\{\varphi\left(\frac{x}{k}\right); \ k \geq k_0 > 0\}$ is bounded in $\mathcal{F}$ for each $\varphi \in \mathcal{F}$. Suppose that $S,T \in \mathcal{F}'$, and $\rho(k) = k^\alpha L(k), \ \alpha > -n, \ or \ \alpha = -n$ and $L(k) \to \infty$ as $k \to \infty$.

a) If $S$ has compact support, then
$$\text{w.lim}_{k \to \infty} S(kx)/\rho(k) = 0, \ \text{in} \ \mathcal{F}'.$$

b) If $T = S$ on $\Gamma \cap \{||x|| > R\}$ for some $R > 0$, and $S(kx) \not\sim \rho(k)g(x), \ k \to \infty$, then $T(kx) \not\sim \rho(k)g(x), \ k \to \infty$, as well.

**Proof.** a) For every $\varphi \in \mathcal{F}$
$$\langle S(kx)/\rho(k), \varphi(x) \rangle = \frac{1}{k^n \rho(k)} \left\langle S(x), \varphi\left(\frac{x}{k}\right) \right\rangle.$$
The set $\{\varphi(x/k); \ k \geq k_0 > 0\}$ is a bounded set in $\mathcal{F}$. Consequently, the set $\{(f(x), \varphi(x/k)); \ k \geq k_0 > 0\}$ is bounded in $\mathbb{R}$, too. Since $k^n \rho(k) \to \infty, \ k \to \infty$, the assertion in a) follows.

b) By assumption, $T - S$ has a compact support. Then, by a), for each $\varphi \in \mathcal{F}$
$$\lim_{k \to \infty} \langle T(kx)/\rho(k), \varphi(x) \rangle = \lim_{k \to \infty} \langle (T - S)(kx)/\rho(k), \varphi(x) \rangle$$
$$+ \lim_{k \to \infty} \langle S(kx)/\rho(k), \varphi(x) \rangle$$
$$= \lim_{k \to \infty} \langle S(kx)/\rho(k), \varphi(x) \rangle$$
which proves b). □
Remark. Proposition 2.4 b) asserts that the quasi-asymptotics related to $\rho(k) = k^\alpha L(k)$, $\alpha > -n$, is a local property. If $\alpha < -n$, then the quasi-asymptotics has no longer this property. The next example, in $S'(\mathbb{R})$, illustrates this fact: If $f = \delta^{(m)} + (x^\beta + m^m)$, $m \in \mathbb{N}_0$, $\beta \notin \mathbb{N}$, then $f$ has quasi-asymptotics related to $\rho(k) = k^q$, where $q = \max(-m - 1, -\beta)$ (see Examples 4 and 7 in 2.3). On the other hand, if $\alpha = -1$, then this property depends on $L$. See also Example 5 of 2.3 in relation to Proposition 2.4 b).

Proposition 2.5. Let $m = (m_1, \ldots, m_n) \in (\mathbb{N}_0)^n$ and $T \in F'_{\Gamma}$. If $T(kx) \sim \rho(k)g(x)$, $k \to \infty$ and $x^m \in M_L$ then
\[(kx)^m T(kx) \sim k^{|m|} \rho(k) x^m g(x), \quad k \to \infty.\]

Proof. We have
\[
\lim_{k \to \infty} \left\langle \frac{(kx)^m T(kx)}{k^{|m|} \rho(k)}, \varphi(x) \right\rangle = \lim_{k \to \infty} \left\langle \frac{T(kx)}{\rho(k)}, x^m \varphi(x) \right\rangle = \left\langle g(x), x^m \varphi(x) \right\rangle = \langle x^m g(x), \varphi(x) \rangle, \quad \varphi \in F. \quad \square
\]

Remark. If $g$ is a distribution with support $\{0\}$, then $t^m g(t)$ may be equal zero.

One of the most useful and frequently used theorems that characterize the quasi-asymptotic behavior is the following one.

Theorem 2.1. Let $T \in S'_\Gamma$ and $\Gamma$ be a regular cone. Then, $T$ has quasi-asymptotics in $\Gamma$ related to $\rho$ if and only if there exists $\alpha \in \mathbb{R}_+$ such that $T^{(-\alpha)}$ has the quasi-asymptotics in $\Gamma$ related to $k^{\alpha} \rho(k)$.

Proof. By the mentioned properties of $\theta_{\alpha, \Gamma}$,
\[
\frac{1}{k^{\alpha} \rho(k)} \langle T^{(-\alpha)}(kx), \varphi(x) \rangle = \frac{k^{-n(\alpha + 1)}}{\rho(k)} \langle T^{(-\alpha)}(x), \varphi(x/k) \rangle
= \frac{k^{-n(\alpha + 1)}}{\rho(k)} \langle T(x), (\theta_{\alpha, \Gamma}(\tau), \varphi((x + \tau)/k)) \rangle
= \frac{1}{\rho(k)} \langle T(kx), (\theta_{\alpha, \Gamma}(\tau), \varphi(x + \tau)) \rangle,
\]
where \( \varphi \in S(\Gamma) \). Since the cone \( \Gamma \) is regular, the mapping: \( \varphi(x) \to \langle \theta_{\alpha, \Gamma}(t), \varphi(x + t) \rangle \) is an automorphism of the space \( S(\Gamma) \). In order to complete the proof, it is enough to take limit in the last equality. \( \square \)

The same theorem can be proved in the same way for \( S' \) if (M.1), (M.2)' and (M.3)' are satisfied.

We now give a structural theorem for quasi-asymptotics on cones.

**Theorem 2.2.** Suppose that \( \Gamma \) is a regular cone. Then \( T \in S'_\Gamma \) has quasi-asymptotics in \( \Gamma \) related to \( \rho \) if and only if there exists an integer \( N \) such that \( T(-N) \) is a continuous function and has the asymptotic behavior in \( \Gamma \) related to \( k_n N \rho(k) \) (see Definition 2.3).

**Proof.** The sufficiency follows from Theorem 2.1. Let us prove the necessity.

Since \( T(kx)/\rho(k) \) converges to \( g \) in \( S' \) (in \( S'_\Gamma \)), there exists \( p \in \mathbb{N}_0 \) such that it converges in \( S'_p(\Gamma) \), as well (see 0.5.1). Also, by properties 4) and 5) of \( \theta_{\alpha, \Gamma} \) there exists \( N_0 \in \mathbb{N} \) such that for every \( t \in \mathbb{R}^n \) and \( N \geq N_0 \) the function \( \theta_{N, \Gamma}(t - x) \in S'_p(\Gamma) \). Then \( T(-N) \) is continuous and

\[
\lim_{k \to \infty} \frac{1}{k_n N \rho(k)} T(-N)(kx) = \lim_{k \to \infty} \frac{1}{k_n N \rho(k)} (T(t), \theta_{N, \Gamma}(kx - t)) = \lim_{k \to \infty} (T(kt)/\rho(k), \theta_{N, \Gamma}(x - t)) = \langle g(t), \theta_{N, \Gamma}(x - t) \rangle.
\]

By property 6) of \( \theta_{\alpha, \Gamma} \), there exist \( A \) and \( q > 0 \) such that

\[
\left| \frac{1}{k_n N \rho(k)} T(-N)(kx) \right| \leq C \| \theta_{N, \Gamma}(x - t) \|_{p, \Gamma} \leq A(1 + \|x\|)^q, \; x \in \Gamma. \quad (2.7)
\]

In this way we proved that \( T(-N) \) has the asymptotic behavior in \( \Gamma \) related to \( k_n N \rho(k) \) (cf. Definition 2.3). \( \square \)

We shall quote a remark given in [192]:

"In many cases it is important to know exactly which primitive of a distribution \( f \), having quasi-asymptotics, already has some asymptotics. There is no simple or universal criterion. For instance, it would be natural to suppose that the condition \( f \in S'_p(\Gamma) \) guarantees the existence of such an \( N \) (depending perhaps on \( p \), on a family \( \{U_k, k \in I\} \) and on a function \( \rho(k), k \in I \)) for which the primitive \( f(-N) \) has the asymptotics."
The following example shows that it is not the case, in general:

Let $\Gamma = \mathbb{R}_+$ and let $
abla \{f_n(\xi) = H(\xi) \exp(i\xi^{1/n}); \ n = 1, 2, \ldots \}$ be a family of functions in $S'$. Each function $f_n$ has the quasi-asymptotics $-i\nu(n)! \delta(\xi)$ over the family of transforms $\{U_k(\xi) = k\xi; \ k > 0\}$ related to the function $\rho(k) = 1/k$, $k > 0$ (cf. Definition 2.1). For every $n \in \mathbb{N}$ the function $f_n(\xi)$ has the asymptotics over the family of transforms $\{U_k(\xi) = k\xi, k > 0\}$ with respect to the function $k^{N-1}$, $k > 0$ if and only if $N - 1 > ((n - 1)/n)N$; that is, when $N > n$ and, hence, $N \to \infty$, as $n \to \infty$. On the other hand, all the functions $f_n$, $n = 1, 2, \ldots$ are infinitely differentiable for $\xi > 0$ and uniformly bounded (cf. [192]).

We prove now that the space of tempered distributions is naturally related to the concept of quasi-asymptotic behavior at infinity.

**Theorem 2.3.** Let $T \in D'$, supp $T \subset \Gamma$, where $\Gamma$ is regular. If

$$\lim_{k \to \infty} \left\langle \frac{T(kx)}{\rho(k)}, \phi(x) \right\rangle = \langle g, \phi \rangle, \ \phi \in D$$

(2.8)

for some regularly varying function $\rho$ and $g \neq 0$, then $T \in S'_{\Gamma}$, and it has quasi-asymptotic behavior at infinity related to $\rho$.

**Proof.** By property 5) of $\theta_{\alpha, \Gamma}$ there exists a sufficiently large $R > 0$ such that $\theta_{m, \Gamma}(e - x) = 0$ for $e \in \text{pr} \Gamma$, $x \in \Gamma$ and $\|x\| > R$. Let $\eta \in D$ be such that $\eta(x) = 1$ for $\|x\| \leq R$. Then, $(\eta(x)T(kx))/\rho(k), k \in \mathbb{N}$, is convergent in $S'_{\Gamma}$. Hence there exists a $p \in \mathbb{N}$ such that it converges in $S'_{\rho}(\Gamma)$ (see 0.5.1). By properties 4) and 5) of $\theta_{\alpha, \Gamma}$, we can find a sufficiently large $m \in \mathbb{N}$ such that

$$\lim_{k \to \infty} \frac{1}{\rho(k)}(\eta(x)T(kx), \theta_{m, \Gamma}(e - x)) = g_m(e),$$

where $\theta_{m, \Gamma}(e - x)$ is in $S'_{\rho}(\Gamma)$. Now the left-hand side can be written without $\eta$ since $\theta_{m, \Gamma}(e - x) = 0$ for $|x| \geq R$, and this implies that

$$\lim_{k \to \infty} \frac{1}{\rho(k)}T^{(-m)}(ke) = \lim_{k \to \infty} \frac{1}{\rho(k)}(T(kx), \theta_{m, \Gamma}(e - x)) = g_m(e)$$

(2.9)

and the last limit exists.

As in Theorem 2.2, it follows that $T^{(-N)} \in L_{loc}^1$ and that the limit in (2.5) for $f = T^{(-N)}$ and with $k^{Nm}\rho(k)$ instead $\rho(k)$ exists. Also (2.6) is satisfied.
Consequently, $T^{(-N)}$ is a locally integrable function having the asymptotic behavior related to $k^n N_p(k)$ in the cone $\Gamma$. Now, we can use Theorem 2.2 to complete the proof. \hfill \Box

We now discuss some results in the context of ultradistributions.

**Theorem 2.4.** Suppose that $M_p$ satisfies (M.1), (M.2) and (M.3). If $f \in E'(\mathbb{R})$ and $\text{supp} f \subset [0, \infty)$, then there exists a $p \in \mathbb{N}_0$ such that
\[
\lim_{k \to \infty} \langle k^{p+1} f(kx), \varphi(x) \rangle = C, \quad \varphi \in S'_{(0, \infty)},
\]
where $C$ can be zero, too.

**Proof.** By Theorem 10.3 in [79], there exist an ultradifferential operator $P(D) = \sum_{n=0}^{\infty} a_n D^n$ of $*$ class and a compactly supported continuous function $G$, $\text{supp} G = K \supset \text{supp} f$, such that
\[
f = \sum_{n=0}^{\infty} a_n D^n G.
\]

Suppose that $a_p \neq 0$ and $a_n = 0$, $n < p$. We have to analyze
\[
\langle k^{p+1} f(kx), \varphi(x) \rangle = \left\langle k^{p+1} \left( \sum_{n=0}^{\infty} a_n D^n G \right)(kx), \varphi(x) \right\rangle
\]
\[
= k^p \sum_{n=p}^{\infty} (-1)^n a_n k^{-n} \left\langle G(x), \varphi^{(n)} \left( \frac{x}{k} \right) \right\rangle = (-1)^{p} a_p \int K \varphi^{(p)} \left( \frac{x}{k} \right) dx
\]
\[
+ \sum_{n=p+1}^{\infty} (-1)^{n} a_n k^{-n+p} \int K G(x) \varphi^{(n)} \left( \frac{x}{k} \right) dx, \quad \varphi \in S'_{(0, \infty)}.
\]

Since
\[
\int K \varphi^{(p)} \left( \frac{x}{k} \right) dx \to \int K G(x) dx \varphi^{(p)}(0), \quad k \to \infty,
\]

it remains to prove that
\[
\left| \sum_{n=p+1}^{\infty} (-1)^n a_n k^{-n+p} \int K G(x) \varphi^{(n)} \left( \frac{x}{k} \right) dx \right| \leq C \sum_{n=p+1}^{\infty} k^{-n+p} \int K |G(x)| \frac{L^n}{M^n} dx
\]
\[
\leq C_1 \sum_{n=p+1}^{\infty} k^{-n+p} \int K |G(x)| dx \to 0, \quad k \to \infty,
\]
Theorem 2.5. Suppose that (M.1), (M.2) and (M.3) are satisfied. Let $T \in \mathcal{D}'(\mathbb{R})$, supp $T \subset [0, \infty)$ and $c(k) = k^\alpha L(k)$, $k > 0, \alpha > -1$. If $T(x + h) \sim c(h)U$, then $T \in \mathcal{S}'_+$ and $T$ has the quasi-asymptotics related to $c$, as well.

Proof. Without loss of generality, we can suppose that $c$ is continuous in $(0, \infty)$ (cf. Remarks after Proposition 1.2).

Denote by $\omega$ a function belonging to $\mathcal{D}'$ such that $\omega(x) = 1$, $x \in [0, h_0]$, $h_0 > 0$. Then, $T = \omega T + (1 - \omega)T$. The support of $\omega T$ is compact, therefore $\omega T \in \mathcal{E}'^*$. By Theorem 2.4

$$\lim_{k \to \infty} \left\langle \frac{\omega(kt)T(kt)}{c(k)}, \phi(t) \right\rangle = 0, \quad \phi \in \mathcal{S}'_{[0, \infty)}.$$

The S-asymptotics is a local property if $\alpha > -1$ (Theorem 1.2, c)). Therefore $(1 - \omega)T$ has the same S-asymptotics as $T$ and supp $(1 - \omega)T \subset [h_0, \infty)$.

We shall use Theorem 1.10 with the assumptions that the set $A$ of that theorem satisfies $A + \mathbb{R}_+ \subset [h_0, \infty)$, $h_0 > 0$, $(\Gamma = \mathbb{R}_+)$ and that supp $f_i \subset (h_0 - \epsilon, \infty), h_0 - \epsilon > 0$, $i = 1, 2$.

Denote by $g_i(h) = f_i(h)/c(h)$, $h \in (h_0 - \epsilon, \infty)$. Then $f_i(h) = c(h)g_i(h), h \geq h_0 - \epsilon > 0$, $i = 1, 2$, and $\lim_{h \to \infty} g_i(h) = C_i, i = 1, 2$. Therefore,

$$(1 - \omega)T(x) = P(D)c(x)g_1(x) + c(x)g_2(x), \quad x > h_0 > 0.$$

It follows that $(1 - \omega)T$ belongs to $\mathcal{S}'_+$. Consequently, $T \in \mathcal{S}'_+$. Suppose now that $\phi \in \mathcal{S}'_{[0, \infty)}$. Then

$$\lim_{k \to \infty} \left\langle \frac{(1 - \omega(kt))T(kt)}{c(k)}, \phi(t) \right\rangle = \lim_{k \to \infty} \left\{ \left\langle \frac{c(kt)g_1(kt)}{c(k)}, \sum_{i=1}^{\infty} \frac{(-1)^i}{k^i} a_i \left( \frac{d}{dt} \right)^i \phi(t) \right\} + \left\langle \frac{c(kt)}{c(k)} (a_0 g_1(kt) + g_2(kt)), \phi(t) \right\rangle \right\}.$$

(2.10)
2. Quasi-asymptotics in $F^\prime$

By the properties of $g_1$ and $g_2$, the last summand in (2.10) has a limit for every $\phi \in S^*_\infty$. More precisely,

$$\lim_{k \to \infty} \left\langle \frac{c(kt)}{c(k)} (a_0 g_1(kt) + g_2(kt)), \phi(t) \right\rangle = \left\langle (a_0 C_1 + C_2)x^\alpha, \phi(x) \right\rangle.$$ 

Since $\left\{ \frac{c(k\gamma)g_1(k)}{c(k)}, k \geq 1 \right\}$ is a bounded set in $S^*_\infty$ and since

$$\sum_{i=1}^{\infty} \left( -\frac{1}{k} \right)^i a_i \left( \frac{d}{dx} \right)^i \phi(x)$$

tends to zero in $S^*_\infty$ as $k \to \infty$, the first summand in (2.10) tends to zero as $k \to \infty$.

From the existence of the $S$-asymptotics of $T$ related to $c$, it follows that $a_0 C_1 + C_2 \neq 0$.

Thus, we have proved that $T \in S^*_+\infty$ and that $T$ has the quasi-asymptotics related to $c$. 

Let $\varphi \in S^*_\infty$ and let $f_\gamma$, $\gamma \in \mathbb{R}$, be the function defined in 0.4. Then $(f_\gamma \ast \varphi)(\xi) = \psi(\xi)$, $\xi \geq 0$ belongs to $S^*_\infty$. One can prove that the mapping $\varphi \mapsto f_\gamma \ast \varphi$ is an automorphism of $S^*_\infty$. The method for proving this property of $f_\gamma$ is the same as for the space $S[0, \infty]$.

Let $f \in S^*_\infty$. Recall, we denote by $f^{(-m)}$ an element belonging to $S^*_\infty$ defined by

$$\left\langle f^{(-m)}(x), \varphi(x) \right\rangle = \left\langle f(x), (f_\gamma \ast \varphi)(x) \right\rangle, \quad \varphi \in S^*_\infty.$$

**Proposition 2.6.** Suppose that $f \in S^*_\infty$ and that $\gamma$ is a real number. Then, $f$ has quasi-asymptotics related to $\rho$ if and only if $f^{(-}\gamma)$ has the quasi-asymptotics related to $k^\gamma \rho(k)$.

**Proof.** By the definition of $f^{(-}\gamma)$,

$$\left\langle f^{(-}\gamma)(x), \varphi(x) \right\rangle = \left\langle f(x), (f_\gamma \ast \varphi)(x) \right\rangle$$

and this exists for every $\varphi \in S^*_\infty$. Therefore, for every $\varphi \in S^*_\infty$ and $k > 0$, we have

$$\left\langle f^{(-}\gamma)(k\xi), \varphi(\xi) \right\rangle = \frac{1}{k} \left\langle f^{(-}\gamma)(\xi), \varphi\left( \frac{\xi}{k} \right) \right\rangle$$

$$= \left\langle \frac{k^\gamma}{k} f(\xi), (f_\gamma \ast \varphi)\left( \frac{\xi}{k} \right) \right\rangle$$

$$= k^\gamma(f(k\xi), (f_\gamma \ast \varphi)(\xi)).$$
Since the mapping: \( \varphi \to \tilde{f}_1 \ast \varphi \) is an automorphisms of \( S^*_0(0,\infty) \), this completes the proof. \( \square \)

**Theorem 2.6.** Suppose that (M.1), (M.2) and (M.3) are satisfied and that \( c(h) = h^\alpha L(h), \ h > h_0, \ \alpha \leq -1. \) If \( T \in D'\ast(R), \ T(x+h) \sim c(h)U, \ h \in R_+ \) and \( \supp T \subset [0,\infty) \), then \( T \in S^*_+ \).

If \( \alpha = -1 \) and \( \hat{L}(x) = \int_{h_1}^x t^{-1}L(t)dt \to \infty, \) as \( x \to \infty, \) then \( T \) has quasi-asymptotics related to \( t^{-1}\hat{L}(t). \) In the other cases, \( T \) has the quasi-asymptotics related to \( t^{-p}, \ p \in N, \) but the limit may be zero.

**Proof.** Let \( w \) be the function used in Theorem 2.5. Then \( T = wT + (1-w)T. \) Since \( wT \) has a compact support, by Theorem 2.4, \( wT \) has quasi-asymptotics related to \( c(k) = k^{-n} \) for an \( n \in N. \)

The support of \( (1-w)T \) belongs to \( [h_0,\infty), \ h_0 > 0. \) Since the \( S \)-asymptotics is a local property, \( (1-w)T \) has the \( S \)-asymptotics related to \( c, \) as well, and

\[
((1-w)T)(t) = \sum_{i=0}^{\infty} a_i D^i(t^\alpha L(t)(t_1^a_1 E_1(t) + (t^a L(t)E_2(t))), \ t > h_0, \quad (2.11)
\]

where \( \supp E_i \subset (h_1,\infty), \ 0 < h_1 \leq h_0 \) and \( \lim_{t \to \infty} E_i(t) = C_i, \ i = 1, 2. \) We know that \( a_0C_1 + C_2 \neq 0 \) because \( (1-w)T \) has the \( S \)-asymptotics related to \( h^\alpha L(h). \)

We can choose \( h_0 \) and \( h_1 \) in such a way that \( a_0E_1(t) + E_2(t) \) does not change the sign when \( t \in (h_1,\infty). \) Using (2.11) and the convolution with \( f_1 \) (cf. 0.4) it follows

\[
((1-w)T)^{(-1)} = f_1 * (t^\alpha L(t)(a_0E_1(t) + E_2(t)))
\]

\[
\quad + \sum_{i=1}^{\infty} a_i D^i(f_1 * (t^\alpha L(t)E_1(t))).
\]

We have to analyze the function \( F = f_1 * (t^\alpha L(t)(a_0E_1(t) + E_2(t))). \) This function is equal to zero in \( (0,h_1) \) and

\[
F(x) = \int_{h_1}^x t^\alpha L(t)(a_0E_1(t) + E_2(t))dt, \ x \geq h_1.
\]
2. Quasi-asymptotics in \( F' \)

Case 1. Let \( \alpha < -1 \) or \( \alpha = -1 \) and \( \int_{h_1}^{\infty} x^{-1} L(x) \, dx < \infty \). Then,

\[
F(x) \to \int_{h_1}^{\infty} t^n L(t)(a_0 E_1(t) + E_2(t)) \, dt, \quad \text{as} \quad x \to \infty.
\]

Consequently, \( F \) has S-asymptotics related to 1. Since \( \int_{h_1}^{\infty} t^{-1} L(t) \, dt < \infty \), \( ((1 - w)T)^{(-1)} \) has S-asymptotics related to 1, too. By Theorem 2.5, \( ((1 - w)T)^{(-1)} \) has quasi-asymptotics related to 1, and by Proposition 2.6, \( (1 - wT) \) has quasi-asymptotics related to \( h^{-1} \).

Case 2. Let \( \alpha = -1 \) and \( \int_{h_1}^{\infty} t^{-1} L(t) \to \infty \), as \( x \to \infty \). Then, \( F(x) = \hat{L}(x) \), where \( \hat{L} \) is also a slowly varying function (see Proposition 1.5.9 in [9]); \( f \) has S-asymptotics related to \( \hat{L}(x) \). We have the same situation with \( \int_{h_1}^{\infty} t^{-1} L(t) \, dt = \hat{L}_1(x) \). By Theorem 2.5, \( ((1 - w)T)^{(-1)} \) has quasi-asymptotics related to \( L_0 = \max(\hat{L}, \hat{L}_1) \) and by Proposition 2.6, \( (1 - wT) \) has quasi-asymptotics related to \( k^{-1}L_0(k) \).

Taking care of the quasi-asymptotics of \( wT \) and \( (1 - w)T \), we have established the assertion of Theorem 2.6. \( \square \)

The following example illustrates different possibilities in case \( \alpha < -1 \).

We use three functions

\[
f(x) = \begin{cases} 
0, & 0 \leq x \leq 1 \\
x^{-1-\varepsilon}, & 1 < x, \ 1 < \varepsilon < 2 
\end{cases}
\]

and \( F(x) = H(x)H(1-x), \ x \in \mathbb{R} \); \( H \) is the Heaviside function.

The quasi-asymptotics are given by:

\[
\langle f(kx), \phi(x) \rangle = k^{-1-\varepsilon} \int_{1/k}^{a} x^{-1-\varepsilon} \phi(x) \, dx
\]

\[
= \frac{1}{\varepsilon} \phi \left( \frac{1}{k} \right) \frac{1}{k} - \frac{1}{\varepsilon} \frac{1}{1-\varepsilon} \phi' \left( \frac{1}{k} \right) \frac{1}{k^2} - \frac{1}{\varepsilon} \frac{1}{1-\varepsilon} \frac{1}{k^{1+\varepsilon}} \int_{1/k}^{a} x^{-1-\varepsilon} \phi''(x) \, dx, \ k > 0;
\]

\[
k \langle F(kx), \phi(x) \rangle = k \int_{0}^{1/k} \phi(x) \, dx = \int_{0}^{1} \phi \left( \frac{t}{k} \right) \, dt \to \phi(0), \ k \to \infty;
\]


\[ k^2 \langle (DF)(kx), \varphi(x) \rangle = k \langle F(kx), \varphi'(x) \rangle \to \varphi'(0), \quad k \to \infty, \quad \varphi \in S'_+ \, . \]

Take now the distribution \( T = f + C_1 F + C_2 DF \). For appropriated constants \( C_1 \) and \( C_2 \), \( T \) can have the quasi-asymptotics related to \( k^{-1}, k^{-2}, k^{-1-\varepsilon} \).

The following problem is discussed in [38]: Let \( f \in S'_+ \) and let \( \{ \varphi_k \}_{k \in \mathbb{N}} \) be a sequence in \( S \). Assume that the following limit exists

\[
\lim_{k \to \infty} \frac{1}{\rho(k)} \langle f(kt), \varphi_k(t) \rangle = c,
\]

where \( \rho \) is a regularly varying function.

The question is to find conditions under which the limit

\[
\lim_{k \to \infty} \frac{1}{\rho(k)} \langle f(kt), \varphi(t) \rangle = c \varphi
\]

exists for all \( \varphi \in S \), i.e., that \( f \in S'_+ \) has the quasi-asymptotics related to \( \rho \).

The authors proved that \( \{ \varphi_k(t) \}_{k \in \mathbb{N}} \) cannot belong to \( S \). But they constructed a new space \( S^a_{b,N,\delta} \) of test functions and the space \( (S^a_{b,N,\delta})' \) of distributions in which the asked question has a positive answer. (cf. Theorem 7 in [38]).

The relation between the spaces \( S_+ \) and \( S^a_{b,N,\delta} \) is given by the following projective limits:

\[
\bigcap_{M, M \in \mathbb{Z}_+} S^a_{b,N,\delta} = S^a_{b,\infty} \equiv S^a_b, \quad \bigcap_{a, b \in \mathbb{R}} S^a_b = S_+ .
\]

The results of Theorem 7 in [38] can be applied in the analysis of quasi-asymptotics of solutions to differential equations, and to Abelian and Tauberian type theorems for integral transforms of distributions.

### 2.3 Quasi-asymptotic behavior at infinity of some generalized functions

We examine first the case when a regular distribution \( T \) is defined by a locally integrable function \( F \) on \( \mathbb{R} \) and has support in an interval \([a, \infty)\), \( a > 0 \). We write \( T = H(x - a) F \), where \( H \) is the Heaviside function.

1. Let \( T = H(x - a) F(x), \quad x \in \mathbb{R}, \) for \( a > 0 \), where \( F \) is a locally integrable function satisfying \( \int_a^\infty |F(x)| dx < \infty \). Then \( T \) has the quasi-asymptotics related to \( k^{-1} \).
2. Quasi-asymptotics in $F'$

It follows at once from
\[
\lim_{k \to \infty} \langle kT(kx), \phi(x) \rangle = \lim_{k \to \infty} \int_{a}^{\infty} F(x) \phi \left( \frac{x}{k} \right) dx = \langle C\delta, \phi \rangle,
\]
where $C = \int_{a}^{\infty} F(x) dx$.

We obtain special cases if either
\[
F(x) \sim x^\alpha L(x) \quad \text{as} \quad x \to \infty \quad \text{and} \quad \alpha < -1
\]
or
\[
F(x) \sim L(x)/x, \quad x \to \infty,
\]
provided that $\int_{a}^{\infty} |L(x)/x| dx < \infty$.

2. Let $a > 0$ and $T(x) = F(x-a)H(x-a)$, $x \in \mathbb{R}$, where $F$ is a locally integrable function such that $F(x) \sim x^\alpha L(x)$ as $x \to \infty$. If
\[
L^*(x) := \int_{a}^{x} \frac{L(t)}{t} dt, \quad x > a
\]
diverges to infinity as $x \to \infty$, then $T$ has the quasi-asymptotics related to $L^*(k)/k, k \to \infty$.

$L^*$ is also slowly varying at infinity (see 0.3 and [9]). Let
\[
G(x) := (H * T)(x) = \int_{a}^{x} F(t) dt, \quad x \in \mathbb{R}.
\]
Since
\[
\lim_{x \to \infty} \frac{G(x)}{L^*(x)} = \lim_{x \to \infty} \frac{F(x)}{L(x)/x} = 1
\]
and $\frac{d}{dx} G = T$, the Structural Theorem 2.2 implies the claim.

3. Let $a > 0$ and $T(x) = F(x-a)H(x-a)$, $x \in \mathbb{R}$, where $F$ is a locally integrable function such that $F(x) \sim x^\alpha L(x)$ as $x \to \infty$ for $\alpha > -1$. Then $T$ has the quasi-asymptotics related to $\rho(k) = k^\alpha L(k)$, $k \to \infty$.

It is obvious, since $G = (H * T)$ is a continuous function on $\mathbb{R}$ such that $G(x) \sim (x^{\alpha+1}/(\alpha+1)) L(x)$, as $x \to \infty$.

Now we give the quasi-asymptotics of some distributions.

4. Denote by $\delta(x-a), a \geq 0$, the delta distribution with support in $a$. Then $\delta^{(m)}(kx-a) \sim k^{-m-1}\delta^{(m)}(x), k \to \infty$.

Indeed, observe that
\[
\langle \delta(kx-a), \phi(x) \rangle = \frac{1}{k} \phi \left( \frac{a}{k} \right) = \frac{\phi(0) + O(1/k)}{k} \sim \frac{\phi(0)}{k}, k \to \infty, \phi \in S;
\]
therefore, $\delta(kx-a) \sim k^{-1}\delta(x)$, $k \to \infty$. The result now follows by differentiating $m$-times the last quasi-asymptotics (cf. Theorem 2.1).

5. For every $S \in \mathcal{E}' \cap \mathcal{S}_{+}$ there exists a natural number $p$ such that $S$ has the quasi-asymptotics related to $k^{-p}$, $k \to \infty$.

For a given $S \in \mathcal{E}' \cap \mathcal{S}_{+}$ there exists $m \in \mathbb{N}_{0}$ and a continuous function $G$ on $\mathbb{R}$ with supp $G \subset [0, \infty)$ such that $S = D^{m}G$ ($D = \frac{d}{dx}$). If supp $S \subset [0, a]$, $a \geq 0$, then we have that $G$ is equal to some polynomial of the order $\leq m-1$ on the interval $(a, \infty)$. Thus for some $0 \leq q \leq m-1$ and some $C \neq 0$

$$G(x) \sim Cx^{q} \quad \text{as} \quad x \to \infty.$$  

This implies that $G$ has the quasi-asymptotics related to $k^{q}$, $k \to \infty$. The Structural Theorem 2.2 implies that $S$ has the quasi-asymptotics related to $k^{q-m}$, $k \to \infty$; in fact $S(kx) \sim Ck^{q-m}\delta^{(m-q-1)}(x)$.

6. Let $F$ be a locally integrable function on $\mathbb{R} \setminus \{0\}$ equal to zero outside of some interval $[0, a]$, $a > 0$, such that

$$F(x) \sim x^{\alpha}L(x) \quad \text{as} \quad x \to 0^{+},$$

where $\alpha \leq -1$ and $L$ is a slowly varying function at zero.

This function can be identified with a distribution $S$ defined by

$$\langle S, \phi \rangle := \int_{0}^{a} F(x)(\phi(x) - \phi(0) - \cdots - \frac{x^{m-1}}{(m-1)!}\phi^{(m-1)}(0))dx \quad (2.12)$$

if $-(m+1) < \alpha \leq -m$, $m \in \mathbb{N}$ and $\phi \in \mathcal{S}(\mathbb{R})$ (see [[138], p. 13]).

The distribution $S$ defined by (2.12) has quasi-asymptotics related to $1/k^{m+1}$, $k \to \infty$. Let us prove this. For $\phi \in \mathcal{S}(\mathbb{R})$, we have

$$\langle k^{m+1}S(kx), \phi(x) \rangle = k^{m} \int_{0}^{a} F(x)\left(\phi\left(\frac{x}{k}\right) - \phi(0) - \cdots - \frac{x^{m-1}}{k^{m-1}(m-1)!}\phi^{(m-1)}(0)\right)dx$$

$$= k^{m} \int_{0}^{a} F(x)\left(\frac{1}{m!}\left(\frac{x}{k}\right)^{m}\phi^{(m)}\left(\frac{\xi x}{k}\right)\right)dx, \quad 0 < \xi x < x,$$

hence

$$\lim_{k \to \infty} \langle k^{m+1}S(kx), \phi(x) \rangle = \frac{(-1)^{m}}{m!}\langle \delta^{(m)}, \phi \rangle \int_{0}^{a} x^{m}F(x)dx, \quad \text{as} \quad x \to \infty.$$
2. Quasi-asymptotics in $F'$

Let us remark that if $\alpha > -1$, then the distribution $S$ given by $F$ is regular and $S(kx) \overset{L}{\sim} Ck^{-1}\delta$, $k \to \infty$, with $C = \int_0^a F(x)dx$.

7. Let $L$ be slowly varying at zero and at infinity.

The distribution $R(x) = (x^\alpha L(x))_+$ (see [138]) has the quasi-asymptotics related to $\rho(x) = x^\alpha L(x)$, $x \to \infty$, if $\alpha \notin \mathbb{Z} = \{-1, -2, \ldots\}$ and related to $\rho_1(x) = x^\alpha L^*(x)$ if $\alpha \in \mathbb{Z}$, where $L^*(x) = \int_a^x L(t)dt$, $x \geq a$.

8. Let $f$ denote a measurable function on $\mathbb{R}$ with support in $[0, \infty)$ satisfying

$$f(x) \sim x^\beta L_1(x) \quad \text{as} \quad x \to 0^+,$$

where $L_1$ is a slowly varying function at zero and $-m-1 < \beta \leq -m$. We suppose that $f$ satisfies the following additional condition

$$x^m f(x) \quad \text{is integrable on} \quad (a, \infty), \quad a > 0. \quad (2.14)$$

We denote by $\tilde{f}$ the following distribution in $S'(\mathbb{R})$ defined by $f$:

$$\langle \tilde{f}, \phi \rangle = \int_0^\infty f(x)(\phi(x) - \phi(0) - \cdots - \frac{x^m}{(m-1)!}\phi^{(m-1)}(0))dx, \quad \phi \in S(\mathbb{R}),$$

if $-m-1 < \beta < -m$, and for some $a > 0$, and if $\beta = -m$,

$$\langle \tilde{f}, \phi \rangle = \int_0^\infty f(x)(\phi(x) - \phi(0) - \cdots - \frac{x^{m-2}}{(m-2)!}\phi^{(m-2)}(0)$$

$$- \frac{x^{m-1}}{(m-1)!}\phi^{(m-1)}(0)H(a-x))dx, \quad \phi \in S(\mathbb{R}). \quad (2.16)$$

The distribution $\tilde{f}$ from (2.15), respectively (2.16), has quasi-asymptotics related to $k^{-m-1}$, resp. $k^{-m}$, provided that both (2.13) and (2.14) hold.

9. Let $f$ satisfy (2.13) for $-m-1 < \beta < -m$ and

$$f(x) \sim x^\gamma L_2(x) \quad \text{as} \quad x \to \infty \quad (\gamma < -m),$$

where $L_2$ is slowly varying at infinity. Then $\tilde{f}$ defined by (2.15) ($-m-1 < \beta < -m$), respectively by (2.16) ($\beta = -m$), has quasi-asymptotics related to $k^{-m-1}$ and $k^{-m}$ respectively.
Observe that this quasi-asymptotics does not depend on the functions $L_1$ and $L_2$.

10. Let $f$ satisfy (2.13) for $-m - 1 < \beta < -m$ and

$$f(x) \sim x^\nu L_2(x) \quad \text{as} \quad x \to \infty \quad (\nu > 0), \quad (2.17)$$

where $L_2$ is slowly varying at infinity.

Then $\tilde{f}$ defined by (2.15) has the quasi-asymptotics related to $x^\nu L_2(x)$.

We suppose in the next example, as usual, that $M_p$ satisfies (M.1), (M.2)' and (M.3)'.

The quasi-asymptotics of ultradistributions is a natural extension of the same notion for distributions. Suppose that $f \in S'_0[0, \infty)$; it defines an ultradistribution $\tilde{f} \in S'_0[0, \infty)$. If it has the quasi-asymptotics as a distribution related to $\rho$, then it has also the quasi-asymptotics as an ultradistribution related to $\rho$, and with the same limit.

On the other hand, the next examples show that there exist elements of $S'_0[0, \infty)$ which are not in $S'_0[0, \infty)$ and have the quasi-asymptotics as ultradistributions.

We deal with the Beurling ultradistributions. In the case of Roumieu’s ultradistributions the treatment is the same.

11. Let $P(D) = \sum_{n=0}^{\infty} a_n D^n$ be an ultradifferential operator of class $(M_p)$ and let $a_0 \neq 0$ and $a_i \neq 0$ for infinitely many $i$. Then $P(D) \delta$ is an element of $S'_{(0, \infty)}$ which is not a distribution. We shall show that $(P(D) \delta)(kx) \sim a_0 k^{-1} \delta(x)$.

By Definition 2.2, we have to consider the following limit:

$$\lim_{k \to \infty} k \langle (P(D) \delta)(kx), \varphi(x) \rangle$$

$$= \lim_{k \to \infty} \langle \delta(y), \sum_{n=0}^{\infty} (-1)^n a_n k^{-n} \varphi^{(n)}(y/k) \rangle$$

$$= a_0 \varphi(0) + \lim_{k \to \infty} \sum_{n=1}^{\infty} (-1)^n a_n k^{-n} \varphi^{(n)}(0), \quad \varphi \in S^{(M_p)}_{[0, \infty)}.$$

It remains to show that the last limit equals zero. Since for every $L > 0$

$$\sup_{n} L^n/M_n \sup_{x \in [0, \infty)} |\varphi^{(n)}(x)| < C_1,$$
2. Quasi-asymptotics in $\mathcal{F}'$

it follows
\[
\left| \sum_{n=1}^{\infty} (-1)^n a_n k^{-n} \varphi^{(n)}(0) \right| \leq C_1 \sum_{n=1}^{\infty} k^{-n} L^n / M_n |\varphi^{(n)}(0)|
\]
\[
\leq C_2 k^{-1} \to 0, \quad k \to \infty.
\]

Consequently, $(P(D)\delta)(kx) \equiv (a_0/k)\delta(x)$ as $k \to \infty$ in $\mathcal{S}'_{[0,\infty)}$.

12. There are also distributions which have no quasi-asymptotics as distributions, but they have the quasi-asymptotics as ultradistributions. One of such distributions is the following one

\[
f = \sum_{n=0}^{\infty} \delta^{(n)}(x - e^n) / M_n.
\]

Suppose that there exist $\rho$ and $g \in \mathcal{S}'_{[0,\infty)}$ such that $f(kx) \equiv \rho(k)g(x)$ in $\mathcal{S}'_{[0,\infty)}$. Then, we would have

\[
\lim_{k \to \infty} \frac{1}{\rho(k)} \left( \sum_{n=0}^{\infty} \delta^{(n)}(kx - e^n) / M_n, \varphi(x) \right) = (g, \varphi)
\]

for every $\varphi \in \mathcal{S}_{[0,\infty)}$.

By Borel’s theorem, there exists $\varphi_0 \in \mathcal{S}_{[0,\infty)}$ such that supp $\varphi_0 \subset [e^{-1}, e]$, and the set \{exp($-p^2 - p$)/($\rho(e^p)M_p$)|$\varphi_0^{(p)}(1)$; $p \in \mathbb{N}$\} is not bounded in $\mathbb{R}$. For this $\varphi_0$ and for the subset \{e^p; $p \in \mathbb{N}$\}, we have

\[
\left| \frac{1}{\rho(e^p)} (f(e^p), \varphi_0(x)) \right| = \frac{1}{\rho(e^p)} \sum_{n=0}^{\infty} M_n \exp(np + p) |\varphi_0^{(p)}(e^n - p)|
\]
\[
= \frac{\exp(-p^2 - p)}{\rho(e^p)M_p} |\varphi_0^{(p)}(1)| \to \infty, \quad p \to \infty.
\]

This is in contradiction to our assumption $f(kx) \equiv \rho(k)g(x)$ in $\mathcal{S}'_{[0,\infty)}$.

This distribution has the quasi-asymptotics as ultradistribution; the proof is the same as for $P(D)\delta$.

2.4 Equivalent definitions of quasi-asymptotics at infinity

Let $T \in \mathcal{S}'_+$. We denote by $\mathcal{L}T$ the Laplace transform of $T$: $\mathcal{L}(T)(z) = (T(t), e^{zt})$, $z \in \mathbb{R} + i\mathbb{R}_+$ (cf. [190], Ch. II, Part 9, [189] and [192], Ch. I, Part 2). We collect in the following theorem some equivalent ways to define quasi-asymptotics over the cone $\mathbb{R}_+$. The part c) is a prototype of a
Tauberian characterization for quasi-asymptotics; Tauberian theorems for various integral transforms will be the main subject of 4.2.

**Theorem 2.7.** Let $T \in S'_+$ and $\rho(k) = k^\alpha L(k)$, $k \geq k_0$. The following statements are equivalent:

1. $\lim_{k \to \infty} T(k \cdot) = Cf_{\alpha+1}$, in $S'(\mathbb{R})$, $C \neq 0$.
2. $\lim_{k \to \infty} T(k \cdot + b) = Cf_{\alpha+1}$, in $D'(\mathbb{R})$, $C \neq 0$, $b \in \mathbb{R}$.
3. $A) \lim_{y \to 0^+} \frac{y}{\rho(1/y)} L(T(iy)) = M 
eq 0$;
   
   $B) \lim_{r \to r_0^+, 0 < r < r_0, 0 < \varphi < \pi} \left| \frac{r}{\rho(1/r)} L(T(re^{i\varphi})) \right| \leq \frac{D_1}{\sin m \varphi}$.
4. $\lim_{k \to \infty} \frac{(T \ast \phi)(k \cdot)}{\rho(k)}(k \cdot) = M \phi_{\alpha+1}$, in $D'(\mathbb{R})$, where $M \phi_{\alpha+1} \neq 0$.
5. For every $\phi \in D(\mathbb{R})$ with the property $L\phi_0(0) \neq 0$ such that
   $\lim_{k \to \infty} \frac{(T \ast \phi_0)(k \cdot)}{\rho(k)}(k \cdot) = M \phi_{\alpha+1}$, in $D'(\mathbb{R})$, $M \phi_0 \neq 0$.
6. For a $\delta$-sequence $(\delta_n)$ (cf. 0.4) there is a $C \neq 0$ such that
   $\lim_{k \to \infty} \frac{(T \ast \delta_n)(k \cdot)}{\rho(k)}(k \cdot) = C f_{\alpha+1}$, in $D'(\mathbb{R})$, and uniformly for $n \in \mathbb{N}$.

**Proof.** $a) \Rightarrow b)$. We start with the relation

$$\left< \frac{T(kx + b)}{\rho(k)}, \phi(x) \right> = \left< \frac{T(kx)}{\rho(k)}, \phi\left( x - \frac{b}{k} \right) \right>, \quad \phi \in D(\mathbb{R}).$$

The set $\{\phi(\cdot - b/k); k \geq 1\}$ is bounded in $D'(\mathbb{R})$. By using the equivalence of the weak and strong sequential convergence in $D'(\mathbb{R})$ and the fact that $\phi(\cdot - b/k) \to \phi$, $k \to \infty$, in $D(\mathbb{R})$, we obtain

$$\lim_{k \to \infty} \left< \frac{T(kx + b)}{\rho(k)}, \phi(x) \right> = \lim_{k \to \infty} \left< \frac{T(kx)}{\rho(k)}, \phi\left( x - \frac{b}{k} \right) \right> = (C f_{\alpha+1}, \phi), \quad \phi \in D(\mathbb{R}).$$
By Theorem 2.3 the last limit holds in $S'(\mathbb{R})$, as well.

$b) \Rightarrow a)$ Take $b = 0$ and apply Theorem 2.3.

$a) \Rightarrow c)$ and $c) \Rightarrow a)$ is proved in [192].

c) $\Rightarrow d)$ For a $T \in S'_+, \ T * \phi \in S'_{(a, \infty)}$, $a \in \mathbb{R}$. We shall show that $T * \phi$ satisfies $c)$, it would imply that $T * \phi$ satisfies $a)$ and then $T$ would satisfy $d)$. We have

$$
\lim_{y \to 0^+} \frac{y}{\rho(1/y)} \mathcal{L}T(yi) \mathcal{L}\phi(iy) = \lim_{y \to 0^+} \frac{y}{\rho(1/y)} \mathcal{L}T(yi) \mathcal{L}\phi(0) = M_{\phi_0}.
$$

Moreover, there exist $D_2 > 0$, $m \in \mathbb{N}_0$ and $r_0 > 0$ such that

$$
\left| \frac{r}{\rho(1/r)} \mathcal{L}T(re^{i\varphi}) \mathcal{L}\phi(re^{i\varphi}) \right| \leq \left| \frac{r}{\rho(1/r)} \mathcal{L}T(re^{i\varphi}) \right| \max_{0 \leq r \leq r_0, 0 \leq \varphi < \pi} |\mathcal{L}\phi(re^{i\varphi})| \leq \frac{D_2}{\sin^m \varphi}.
$$

d) $\Rightarrow e)$ It is obvious.

e) $\Rightarrow a)$ The assumptions in $e)$ imply $a)$ and consequently $c)$ for $T * \phi_0$.

Now, we have

$$
\lim_{y \to 0^+} \frac{y}{\rho(1/y)} \mathcal{L}T(iy) \mathcal{L}\phi_0(iy) = \lim_{y \to 0^+} \frac{y}{\rho(1/y)} \mathcal{L}T(iy) \mathcal{L}\phi_0(0) = M \neq 0.
$$

Taking care of the property that $\mathcal{L}\phi_0(0) \neq 0$, we have

$$
\left| \frac{r}{\rho(1/r)} \mathcal{L}T(re^{i\varphi}) \right| \leq \frac{D'}{\sin^{m'} \varphi},
$$

where $0 < r \leq r_0'$, and $0 \leq \varphi < \pi$ for appropriate $r_0'$, $D'$ and $m'$. This gives $c)$ for $T$ and consequently $a)$.

$a) \Rightarrow f)$ Let $\phi \in D(\mathbb{R})$ and $n \in \mathbb{N}$. Then

$$
\left\langle \frac{T * \delta_n(x/k)}{\rho(k)}, \phi(x) \right\rangle = \frac{1}{k \rho(k)} \left\langle T * \delta_n(x), \phi \left( \frac{x}{k} \right) \right\rangle \left\langle \begin{array}{c}
T(x), \left( \delta_n * \phi \left( \frac{\cdot}{k} \right) \right)(x) \\
\int_{\mathbb{R}} \delta_n(-t) \phi \left( x - \frac{t}{k} \right) dt
\end{array} \right\rangle.
$$
The set
\[ \left\{ \int_{\mathbb{R}} \delta_n(-t) \phi \left( x - \frac{t}{k} \right) dt; \ n \in \mathbb{N}, \ k \in (0, \infty) \right\} \]
is bounded in \( \mathcal{D}(\mathbb{R}) \) because of the properties of \( \langle \delta_n \rangle_n \) (cf. 0.4). Therefore, we have
\[ \langle \frac{T(kx)}{\rho(k)}, \int_{\mathbb{R}} \delta_n(-t) \phi \left( x - \frac{t}{k} \right) dt \rangle \to \langle Cf_{\alpha+1}, \phi \rangle, \ k \to \infty, \]
uniformly for \( n \in \mathbb{N} \). Thus, for any \( \varepsilon > 0 \) there is a \( k_0(\varepsilon) \) such that
\[ \left| \langle \frac{T(kx)}{\rho(k)}, \int_{\mathbb{R}} \delta_n(-t) \phi \left( x - \frac{t}{k} \right) dt \rangle - \langle Cf_{\alpha+1}, \phi \rangle \right| \leq \varepsilon, \ k > k_0(\varepsilon); \]
which proves a) \( \Rightarrow \) f).

\( f) \Rightarrow e) \) Since \( \int_{-\infty}^{\infty} \delta_n(x) = 1, \ n \in \mathbb{N} \), we have \( \mathcal{L}\delta_n(0) \neq 0 \). This implies that e) holds. \( \square \)

2.5 **Quasi-asymptotics as an extension of the classical asymptotics**

We have seen in Proposition 2.3 that if a locally integrable function \( f \) has asymptotic behavior related to a function \( \rho \), then \( \tilde{f} \) has the quasi-asymptotics related to \( \rho \). The following theorem also goes in this direction.

**Theorem 2.8.** Let \( \Gamma \) be a closed convex acute solid cone in \( \mathbb{R}^n \), \( f \in S'(\Gamma) \cap L^1_{\text{loc}}(\Gamma \cap \{ \xi \in \mathbb{R}^n; |\xi| > R \}) \) for some \( R > 0 \) and let \( \rho(k) = k^\alpha L(k), \ \alpha > -n. \) If, for any \( e \in \text{pr} \Gamma \) (\( \text{pr} \Gamma = \{ x \in \mathbb{R}^n; x \in \Gamma, |x| = 1 \} \)), the limits
\[ \lim_{|\xi| \to \infty} \frac{1}{\rho(|\xi|)} f(|\xi| e) = g(e) \neq 0 \]

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exist, and for some $g_1 \in L^1(\text{pr}\Gamma)$ the estimate

$$\frac{|f(\xi)|}{\rho(|\xi|)} \leq g_1 \left( \frac{\xi}{|\xi|} \right), \quad |\xi| > R, \; \xi \in \Gamma,$$

is satisfied, then $f$ has quasi-asymptotics related to $\rho$.

The proof is similar to that of Proposition 2.3 (see Theorem 2, Chapter I, §3.3. in [192]).

Remark. The condition $\alpha > -n$ in Theorem 2.8 is essential. This fact is illustrated by the following two examples.

Let $f(x) = H(x - 1)x^{-1-\varepsilon}, x \in \mathbb{R}, \varepsilon > 0$. Then $f(x) \sim x^{-1-\varepsilon}, x \to \infty$, but by Example 1 in 2.3 the regular distribution $f$ has the quasi-asymptotics related to $\rho(k) = k^{-1}$ for any $\varepsilon > 0$.

Let $T(x) = H(x - 2)(x \log^2 x)^{-1}, x \in \mathbb{R}$. Then $T(x) \sim (x \log^2 x)^{-1}, x \to \infty$. But by the same Example 1 in 2.3, $T$ has the quasi-asymptotics related to $\rho(k) = k^{-1}$.

A more difficult question is to find conditions under which the quasi-asymptotics implies the ordinary asymptotics of $f$. A partial answer is given in the ensuing theorem.

**Theorem 2.9.** Let $T \in S'_+$ be equal to a locally integrable function $f$ in some interval $[b, +\infty), b > 0$, with quasi-asymptotic behavior related to $\rho(k) = k^\alpha L(k), \alpha > -1$. If for some $m \in \mathbb{N}$ the function $x^m f(x), x \geq b$, is monotonous, then $f$ has the asymptotic behavior at infinity related to the same regularly varying function $\rho$.

Proof. By Proposition 2.5, the function $g(x) = x^m f(x), x \geq b$, has the quasi-asymptotic behavior at infinity related to $\rho_1(k) = k^{\alpha+m} L(k)$. The monotonicity of $g$ implies that its distributional derivative $Dg$ can be written as $Dg = B + \mu$, where $B \in E'_+$ and $\mu$ is a positive measure with support in $[b, \infty)$. By Theorem 2.1, $Dg$ has the quasi-asymptotics related to $k^{\alpha+m-1} L(k)$. From Proposition 2.4 a), we see that $Dg$ and $\mu$ have the same quasi-asymptotic behavior at infinity, related to $k^{\alpha+m-1} L(k)$, i.e.,

$$\lim_{k \to \infty} \frac{1}{k^{\alpha+m-1} L(k)} (\mu(kx), \phi(x)) = C(f_{\alpha+m}, \phi), \quad \phi \in S(\mathbb{R}). \quad (2.18)$$

Choosing $\phi_\varepsilon$ and $\psi_\varepsilon$ in $\mathcal{D}(\mathbb{R})$ with the properties

$$\phi_\varepsilon(x) = 1, \text{ for } |x| \leq 1 - \frac{\varepsilon}{2} \text{ and } \phi_\varepsilon(x) = 0 \text{ for } |x| \geq 1,$$

...


ψ_ε(x) = 1, for |x| ≤ 1 and ψ_ε(x) = 0 for |x| ≥ 1 + \frac{ε}{2}, for 0 < ε < 1, we obtain
\[
\frac{1}{k^{α+m-1}L(k)}(μ(kx), φ_ε(x)) ≤ \frac{1}{k^{α+m-1}L(x)}(μ(kx), H(1-x)) ≤ \frac{1}{k^{α+m-1}L(k)}(μ(kx)ψ_ε(x)),
\]
(2.19)
since
\[φ_ε(x) ≤ H(1-x) ≤ ψ_ε(x) \text{ for } x > 0.\]
Using (2.18), we see that, both, the left and the right hand side of (2.19) tend to numbers which do not differ more than ε. Thus the expression in the middle of (2.18) tends to some limit independent of ε. Since (H * μ) is equal to g(x) on (b_1, ∞), b_1 ≥ b and
\[
\frac{1}{k^{α+m-1}L(k)}(μ(kx), H(1-x)) = \frac{1}{k^{α+m}L(k)}(H * μ)(k),
\]
we conclude that g has ordinary asymptotic behavior at infinity related to ρ_1, and this implies the statement. □

Remark. It is easy to find a function which has quasi-asymptotics but not an asymptotic behavior. An example is given by \(f(x) = H(x)\sin x, x \in \mathbb{R}; f(kx) \sim k^{-1}\delta(x), k \to \infty.\)

In such a way, the quasi-asymptotics extends the notion of the asymptotic behavior of a locally integrable function.

2.6 Relations between quasi-asymptotics in \(D'(\mathbb{R})\) and \(S' (\mathbb{R})\)

In this section we make a preliminary investigation of some questions raised by Theorem 2.3:

As in Theorem 2.3, suppose that \(T ∈ D'\) satisfies (2.8), with \(g ∈ D'\), but we now remove the assumptions over the support of \(T\) (we allow the support to be any subset of \(\mathbb{R}^n\)). It then is natural to ask: Does \(T ∈ S'\)? Furthermore, does the limit exist in \(S'\)? Let us observe that Theorem 2.3 does not give an answer to such a question because the cone \(Γ\) is acute and it cannot be the whole \(\mathbb{R}^n\).
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The following theorem and its corollary give a partial answer to this question. We will postpone the complete answer (in one dimension) for 2.10.

**Theorem 2.10.** Let $f \in \mathcal{D}'(\mathbb{R})$ and suppose that there exists, in the sense of convergence in $\mathcal{D}'(\mathbb{R})$, the limit
\[
\lim_{k \to \infty} f(kx)/\rho(k) = g(x) \neq 0,
\] (2.20)
where $\rho(k)$, $k \in (0, \infty)$, is a positive continuous function on $(0, \infty)$. Then,

(i) $\rho(x) = x^v L(x), x \in (0, \infty)$, for some $v \in \mathbb{R}$ and some slowly varying function $L$, and $g$ is a homogeneous distribution with degree of homogeneity $v$.

(ii) $f \in \mathcal{S}'(\mathbb{R})$.

(iii) If $v > -1$, then the limit (2.20) exists in the sense of convergence in $\mathcal{S}'(\mathbb{R})$.

(iv) If $v = -1$ and $1/L(x), x \in (a, \infty)$, is bounded, then the limit (2.20) exists in the sense of convergence in $\mathcal{S}'(\mathbb{R})$, as well.

It should be noticed that the proof is quite different from that of Theorem 2.3 where we used the fact that $\mathcal{D}'_\Gamma$ and $\mathcal{S}'_\Gamma$ are convolution algebras. The absence of the convolution algebra structure makes the argument more complex.

**Proof.** (i) Let $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\langle g, \varphi \rangle \neq 0$. We have
\[
\lim_{k \to \infty} \langle f(kmx)/\rho(k), \varphi(x) \rangle = \langle g(mx), \varphi(x) \rangle;
\]
\[
\lim_{k \to \infty} (\rho(mk)/\rho(k))(f(kmx)/\rho(km), \varphi(x))
\]
\[
= \langle g(x), \varphi(x) \rangle \lim_{k \to \infty} (\rho(mk)/\rho(k)), \quad m > 0.
\]
This implies that, for every $m > 0$, we have the existence of the limit
\[
\lim_{k \to \infty} (\rho(mk)/\rho(k)) = d(m).
\]
By [[148], p. 17], we obtain that for some $v \in \mathbb{R}$ and some slowly varying function $L$, $\rho(k)$ and $d(k)$ are of the form
\[
\rho(k) = k^v L(k), \quad d(k) = k^v, \quad k \in (a, \infty).
\]
Since \( \langle g(mx), \varphi(x) \rangle = m^v \langle g(x), \varphi(x) \rangle, \) \( m > 0 \), we obtain that \( g \) is homogeneous of degree \( v \).

(ii) The set \( \{ f(kx)/(k^v L(k)); \ k > 0 \} \) is a bounded subset of \( D'(\mathbb{R}) \).

From Theorem XXII, Chapter VI in [146], it follows that for a given open bounded neighborhood of zero \( \Omega \) there exists a compact neighborhood of zero \( K \) and a non-negative integer \( m \) such that for any \( \varphi \in D_K^m(\mathbb{R}) \)

\[
\Omega \ni x \mapsto ((f(kx)/(k^v L(k))) \ast \varphi(t))(x), \ k \in (0, \infty), \tag{2.21}
\]

is a family of functions which are continuous and uniformly bounded on \( \Omega \). Let \( \Omega = (-2, 2) \) and \( K = [-\varepsilon, \varepsilon] \). Since the weakly bounded family (2.21) is strongly bounded in \( D_K^m(\mathbb{R}) \), we obtain that for every bounded set \( A \subseteq D_{[-\varepsilon, \varepsilon]}^m \), the set of functions

\[
\{ \Omega \ni x \mapsto ((f(kt)/(k^v L(k))) \ast \varphi(t))(x); \ k > 0, \ \varphi \in A \}
\]

is a bounded family of continuous functions on \( \Omega \). Let \( \psi \in D_{[-\varepsilon, \varepsilon]}^m \) and let

\[
\varphi_k(x) = \psi(kx)/k^m, \ \ x \in \mathbb{R}, \ \ k \geq 1.
\]

Since \( \text{supp} \varphi_k(x) \subseteq \{ x; |x| \leq \varepsilon/k \} \subseteq [-\varepsilon, \varepsilon] \), we have that \( A = \{ \varphi_k(x); \ k \geq 1 \} \) is a bounded family in \( D_{[-\varepsilon, \varepsilon]}^m \) and that

\[
\{((f(kt)/(k^v L(k))) \ast \varphi_r(t))(x); \ k > 0, \ r \geq 1 \}
\]

is a bounded family of continuous functions on \( \Omega \). Taking \( r = k \), we obtain that for some \( M > 0 \)

\[
|((f(kt)/(k^v L(k))) \ast (\psi(kt)/k^m))(x)| \leq M, \ x \in (-2, 2), \ k \geq 1.
\]

From

\[
(f(kt) \ast \psi(kt))(x) = (f(kt), \psi(k(x - t))) = k^{-1}(f(t), \psi(kx - t))
\]

we obtain that

\[
| (f \ast \psi)(kx)/(k^{v+m+1} L(k)) | < M \quad \text{for} \quad x \in (-2, 2), \ k \geq 1.
\]

Taking \( x = 1 \) and \( x = -1 \) it follows that for any \( \psi \in D_{[-\varepsilon, \varepsilon]}^m \) there exists \( M_\psi > 0 \) such that

\[
| (f \ast \psi)(x) | \leq M_\psi (1 + |x|^{v+m+1} L(|x|)), \ x \in \mathbb{R}.
\]

By (VI, 6; 22) in [146], we obtain

\[
f = \frac{d^{2n}}{dx^{2n}}(\gamma E \ast f) - \psi \ast f, \tag{2.22}
\]
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where $E$ is the fundamental solution of $d^2 s E/dx^2 s = \delta$, $\gamma \in D_{[-\varepsilon,\varepsilon]}$, $\gamma \equiv 1$ in a neighborhood of zero and $\psi \in D_{[-\varepsilon,\varepsilon]}$. If $s$ is large enough, then $\gamma E \in D_{m [-\varepsilon,\varepsilon]}$. Thus (2.22) implies that $f \in S'(R)$.

(iii) We can rewrite (2.22) in the form $f = d^2 s f_1/dx^2 s + f_2$, where $f_1$ is a continuous and $f_2$ a smooth function such that

$$\sup\{|f_1(x)|, |f_2(x)|\} \leq M(1 + |x|^{v+m+1} L(|x|)),$$

for some $M > 0$.

Since

$$f(kx)/(k^v L(k)) = \left[\left(\frac{d^2 s}{dx^2 s} f_1\right)(kx) + f_2(kx)\right]/(k^v L(k))$$

$$= \frac{d^2 s}{dx^2 s} (f_1(kx))/(k^{v+2} L(k)) + f_2(kx)/(k^v L(k)),$$

we obtain, for $s$ large enough that for any $\varphi \in D(R)$:

$$\lim_{k \to \infty} (f(kx)/(k^v L(k)), \varphi(x)) = \lim_{k \to \infty} (f_2(kx)/(k^v L(k)), \varphi(x)) = (g(x), \varphi(x)).$$

(2.23)

Let us set

$$f_2+(x) = \begin{cases} f_2(x), & x > 0 \\ 0, & x \leq 0 \end{cases}, \quad f_2-(x) = \begin{cases} f_2(x), & x < 0 \\ 0, & x \geq 0 \end{cases}.$$

Clearly, for any $\varphi \in D(0, \infty)$ (supp $\varphi \subset (0, \infty)$),

$$\lim_{k \to \infty} (f_2(kx)/(k^v L(k)), \varphi(x)) = \lim_{k \to \infty} (f_2+(kx)/(k^v L(k)), \varphi(x)).$$

If $\varphi \in D(0, \infty)$, then $\psi(t) = \varphi(e^t)e^t$, $t \in R$, is an element in $D(R)$. Moreover, the mapping $\varphi \mapsto \psi$ defined above is a bijection. Since for $\varphi \in D(0, \infty)$,

$$\int_0^\infty f_2+(kx) \varphi(x) dx = \int_{-\infty}^\infty f_2+(ke^r) \varphi(e^r) e^r dt,$$

by putting $k = e^r$, $r \in R$, we obtain that for a function $F_{1,2}(t) = f_2+(e^t)$, $t \in R$, there exists the limit

$$\lim_{r \to \infty} \langle F_{2,1}(t + r)/(e^{vr} L(e^r)), \psi(t) \rangle.$$
for any $\psi \in \mathcal{D}(\mathbb{R})$. Using again Theorem XXII, Chapter VI in [146], we obtain that for any open set $\Omega \ni 0$ there exists a compact neighborhood of 0 and a non-negative integer $m$ such that for any $\varphi \in \mathcal{D}^m_R(\mathbb{R})$,

$$x \mapsto (F_{2,1}(t + r) * \varphi(t))(x)/(e^{rt}L(e^t)), \quad r \geq 0,$$

is a bounded family of continuous functions on $\Omega$. Since $(F_{2,1}(t + r) * \varphi(t))(x) = (F_{2,1}(t) * \varphi(t))(x + r), x \in \mathbb{R}$ taking $x = 0$ and using (VI; 6; 22) in [146], we obtain

$$F_{2,1} = \frac{d^{2l}}{dx^{2l}}H_{2,1} + G_{2,1},$$

where $H_{2,1}$ is a continuous function and $G_{2,1}$ is a smooth function on $\mathbb{R}$ such that

$$\sup \{|H_{2,1}(t)|, |G_{2,1}(t)|\} < Me^{rt}L(e^t), \quad t > 0.$$

(2.24)

If $t \in (-\infty, 0)$, then $e^t \in (0, 1)$ and since $F_{2,1}(t)$ is bounded on $(-\infty, 0)$, we obtain that $H_{2,1}$ and $G_{2,1}$ are bounded on $(-\infty, 0)$. Namely, both functions are equal to the convolution of $F_{2,1}$ with suitable functions with compact supports.

From

$$f_{2+}(e^t) = \frac{d^{2l}}{dx^{2l}}H_{2,1}(t) + G_{2,1}(t), \quad t \in (-\infty, \infty),$$

we obtain

$$f_{2+}(x) = \sum_{p=1}^{2l} a_p x^p \frac{d^p}{dx^p}(H_{2,1}(\log x)) + G_{2,1}(\log x), \quad x > 0,$$

where $a_p$ are suitable constants.

Set now

$$\tilde{G}_{2,1}(x) = \begin{cases} G_{2,1}(\log x), & x > 0 \\ 0, & x \leq 0 \end{cases}, \quad \tilde{H}_{2,1}(x) = \begin{cases} H_{2,1}(\log x), & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

The distributions $f_{2+} \in \mathcal{D}'(-\infty, \infty)$ and $\sum_{p=1}^{2l} a_p x^p \frac{d^p}{dx^p}H_{2,1} + G_{2,1}$ are equal to each other. For every $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\int_0^\infty f_{2+}(kx)\varphi(x)dx = \langle f_{2+}(kx), \varphi(x) \rangle$$

$$= \sum_{p=1}^{2l} a_p ((kx)^p \frac{1}{k^p})(\tilde{H}_{2,1}(kx))(\varphi(x)) + (\tilde{G}_{2,1}(kx), \varphi(x))$$
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\[ = \sum_{p=1}^{2l} a_p (-1)^p \langle H_{2,1}(kx), (x^p \varphi(x))^{(p)} \rangle + \langle G_{2,1}(kx), \varphi(x) \rangle. \]

The functions $H_{2,1}$ and $G_{2,1}$ are bounded on $(-\infty, 0)$. Thus, by (2.24), we have that for some $M > 0$

\[ \sup \{ |\bar{G}_{2,1}(kx)|, |\bar{H}_{2,1}(kx)| \} \leq M(kx)^{v} L(kx), \quad x > \frac{1}{k}, \quad (2.25) \]

\[ \sup \{ |\bar{G}_{2,1}(kx)|, |\bar{H}_{2,1}(kx)| \} \leq M, \quad 0 < x \leq \frac{1}{k}, \quad (2.26) \]

Since $v > -1$, these inequalities imply that for any $\varphi \in \mathcal{D}(\mathbb{R})$,

\[ \int_{0}^{\infty} f_{2+}(kx) \varphi(x) dx / (k^n L(k)) < \infty. \quad (2.27) \]

In a similar way as in the proof of Theorem 2.3, one can prove that (2.25) holds for every $\varphi \in \mathcal{S}(\mathbb{R})$. If we put $F_{2,2}(t) = f_{2-}(-e^t)$, $t > 0$, then, by the same arguments as above, one can prove that for every $\varphi \in \mathcal{S}(\mathbb{R})$,

\[ \int_{-\infty}^{0} f_{2-}(kx) \varphi(x) dx / (k^n L(k)) < \infty. \quad (2.28) \]

By the Banach–Steinhaus Theorem, it follows from (2.23), (2.27) and (2.28) that

\[ \lim_{k \to \infty} f(kx)/k^n L(k) = g(x) \]

in the sense of convergence in $\mathcal{S}'(\mathbb{R})$.

(iv) The proof is the same as (iii). Namely, in this case the estimates (2.25) and (2.26) imply the claim as well. \[ \square \]

We can extend (ii) in Theorem 2.10 to the case $v \notin -\mathbb{N}$. First, we shall recall two assertions (see Lemma 1 and Lemma 2, §7.4 in [192]).

**Lemma 2.1.** Let $f \in \mathcal{S}'(\mathbb{R})$ and let $\rho$ be a regularly varying function of the degree $\beta \neq -1, -2, \ldots$

a) If there exists the limit

\[ f(kt)/\rho(k) \to g \neq 0, \quad k \to \infty, \quad \text{in } \mathcal{S}'(\mathbb{R}), \]

where $f = f_+ + f_-$ ($f_+ \in \mathcal{S}'_+$ and $f_- \in \mathcal{S}'_-$), then there exists $N \in \mathbb{N}_0$

such that for $f_{+,N} = i^N f_+$ and $f_{-,N} = i^N f_-$,

\[ \lim_{k \to \infty} f_{\pm,N}(kx)/(k^N \rho(k)) = C_{\pm} f_{\beta+N+1}(\pm t), \]
where \((C_+, C_-) \neq (0, 0)\).

b) If \(f \in S'_+\) and if, for \(N \in \mathbb{N}_0\), \(f_N = t^N f\) has quasi-asymptotics at \(\infty\) related to \(k^N \rho(k)\), then:

1) If \(\beta > -1\), then \(f\) has quasi-asymptotics at \(\infty\) related to \(\rho(k), k > 0\).

2) If \(\beta < -1\), then there exist \(a_j \in \mathbb{R}\), \(j = 0, 1, \ldots, p\), such that

\[
g(t) = f(t) + \sum_{j=0}^{p} a_j \delta^{(j)}(t)
\]

has the quasi-asymptotics related to \(\rho(k)\).

**Corollary of Theorem 2.10.** Let \(f \in \mathcal{D}'(\mathbb{R})\). Suppose the following limit exists, in the sense of convergence in \(\mathcal{D}'(\mathbb{R})\),

\[
\lim_{k \to \infty} f(kx)/(k^v L(k)) = g(x) \neq 0,
\]

where \(v \in \mathbb{R} \setminus \{-1, -2, -3, \ldots\}\). Then this limit exists in the sense of convergence in \(S'(\mathbb{R})\) as well.

**Proof.** Theorem 2.10. implies that \(f \in S'(\mathbb{R})\).

Let \(n \in \mathbb{N}\) be such that \(v + n > -1\). Clearly, for \(f_1(x) = x^n f(x), x \in \mathbb{R}\), there holds

\[
\lim_{k \to \infty} (f_1(kx)/(k^v L(k)), \varphi(x)) = (x^n g(x), \varphi(x)), \quad \varphi \in \mathcal{D}(\mathbb{R}).
\]

There exist distributions \(f_+(x) \in S'_+\) and \(f_- \in S'-(\text{supp} f_- \subset (-\infty, 0])\) such that \(f = f_+ + f_-\). This decomposition of \(f\) implies the decomposition of \(f_1\) : \(f_1(x) = x^n f_+(x) + x^n f_-(x), x \in \mathbb{R}\), where \(x^n f_+(x) \in S'_+\) and \(x^n f_-(x) \in S'_-\).

From Lemma 2.1 a), it follows that for some \(m \in \mathbb{N}\),

\[
((m+n) f_\pm(t))(kx)/(k^{m+n+v} L(k)) \to C_\pm f_{v+m+n+1}(x) \quad \text{in} \quad S'(\mathbb{R}) \quad \text{as} \quad k \to \infty,
\]

where \((C_+, C_-) \neq (0, 0)\).

Now, Lemma 2.1 b) implies that for some constants \(a_\alpha, \alpha = 0, 1, \ldots, p\), and \(b_\beta, \beta = 0, 1, \ldots, r\),

\[
(f_+(kt) + \sum_{\alpha=0}^{p} a_\alpha \delta^{(\alpha)}(kt))/(k^v L(k)) \to C_1 f_{v+1}(t) \quad \text{in} \quad S'(\mathbb{R}) \quad \text{as} \quad k \to \infty,
\]

\[
(f_-(kt) + \sum_{\beta=0}^{r} b_\beta \delta^{(\beta)}(kt))/(k^v L(k)) \to C_2 f_{v+1}(-t) \quad \text{in} \quad S'(\mathbb{R}) \quad \text{as} \quad k \to \infty,
\]
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where $(C_1, C_2) \neq (0, 0)$.

Let us notice that in Lemma 2.1 b) it is assumed that the limit distribution $g$ is different from 0. But this assertion also holds for $g = 0$.

Thus for suitable constants $\tilde{C}_\alpha$, $\alpha = 0, 1, \ldots, s$, $s = \max\{p, r\}$

$$(f(kt) + \sum_{\alpha=0}^s \tilde{C}_\alpha \delta^{(\alpha)}(kt))/(k^v L(k)) \to C_1 f_{v+1}(t) + C_2 f_{v+1}(-t),$$

as $k \to \infty$ in $\mathcal{S}'(\mathbb{R})$. The fact that $v \in \mathbb{R} \setminus \mathbb{N}$ implies that

$$\frac{1}{k^v L(k)} \sum_{\alpha=0}^s \tilde{C}_\alpha \delta^{(\alpha)}(kt) \to 0, \quad k \to \infty,$$

and this completes the proof. \hfill \square

2.7 Quasi-asymptotics at $\pm \infty$

In 2.1–2.5 the essential assumption was that the generalized functions were supported by an acute cone $\Gamma$. In this section, we relax this restriction over the support in the one-dimensional case. In fact, we already faced this situation in 2.6 and studied some problems occurring when the distributions are not supported by an acute cone. We will focus in the case of distribution spaces. Let us observe that some of the results below can be also generalized to $\mathcal{F}'(\mathbb{R}^n)$.

Definition 2.4. It is said that $f \in \mathcal{F}'(\mathbb{R})$ has quasi-asymptotics at $\pm \infty$ related to some positive measurable function $c(k)$, $k \in (a, \infty)$, $a > 0$, if for some $g \in \mathcal{F}'(\mathbb{R})$

$$\lim_{k \to \infty} \langle f(kx)/c(k), \phi(x) \rangle = \langle g(x), \phi(x) \rangle, \quad \phi \in \mathcal{F}(\mathbb{R}).$$

We write in short: $f(kx) \overset{q}{\sim} c(k)g(x), \; k \to \infty$ in $\mathcal{F}'(\mathbb{R})$, or simply $f \overset{q}{\sim} g$ at $\pm \infty$ related to $c(k)$.

The results of 2.7 give us already some important properties of quasi-asymptotics at $\pm \infty$, they are stated in the following remark.

Remark. If $g \neq 0$ and $\mathcal{F}'(\mathbb{R}) = \mathcal{D}'(\mathbb{R})$ in Definition 2.4, then, by Theorem 2.10, $f \in \mathcal{S}'(\mathbb{R})$, $c(x) = x^v L(x)$, $x \in (0, \infty)$, for some $v \in \mathbb{R}$ and some slowly varying function $L$, and $g$ is a homogeneous distribution with degree of homogeneity $v$. 
Several properties of the quasi-asymptotics at $\pm \infty$ are listed in the following theorems.

**Theorem 2.11.** Let $f \in \mathcal{D}'(\mathbb{R})$ and $f \overset{L}{\sim} g$ at $\pm \infty$ related to $k^\nu L(k)$. Then:

(i) $f^{(m)} \overset{L}{\sim} g^{(m)}$ at $\pm \infty$ related to $k^\nu-m L(k)$, $k > a$, $m \in \mathbb{N}$;

(ii) if $m \in \mathbb{N}$ then $x^m f \overset{L}{\sim} x^m g$ at $\pm \infty$ related to $k^\nu+m L(k)$;

(iii) if $\phi \in \mathcal{E}(\mathbb{R})$ and $c_1$ is a measurable positive function on some interval $(a, \infty)$, $a > 0$, such that

\[
\frac{\phi(kx)}{c_1(k)} \to \phi_0(x) \quad \text{in} \quad \mathcal{E}(\mathbb{R}), \quad k \to \infty, \quad x \in \mathbb{R},
\]

then $f \phi \overset{L}{\sim} g \phi_0$ at $\pm \infty$ related to $c_1(k)k^\nu L(k)$.

**Proof.** Properties (i) and (ii) follow easily from the definition. For (iii), observe that if $\varphi \in \mathcal{D}(\mathbb{R})$, then $\varphi(x)/(k^\nu L(k)) \to \varphi_0(x)$ in $\mathcal{D}(\mathbb{R})$. By the equivalence between weak and strong sequential convergence in $\mathcal{D}'$, we have

\[
\lim_{k \to \infty} \left< \frac{f(kx)}{c_1(k)k^\nu L(k)}, \varphi(x) \right> = \lim_{k \to \infty} \left< \frac{f(kx)}{k^\nu L(k)}, \varphi(x) \frac{\phi(kx)}{c_1(k)} \right> = \left< g \phi_0, \varphi \right>. \quad \square
\]

**Theorem 2.12.** Let $f \in \mathcal{E}'(\mathbb{R})$ and $f \overset{L}{\sim} g$ at $\pm \infty$ related to $k^\nu L(k)$, $g \neq 0$. Then $L(k) = 1$, $k > a$, $\nu \in \mathbb{N}$, and $g(x) = C\delta(-\nu-1)(x)$, for some constant $C$. Moreover, the limit in Definition 2.4 can be extended on $\mathcal{S}(\mathbb{R})$.

**Proof.** It is the same as that of Example 5 in 2.3. \quad \square

**Theorem 2.13.** Let $F$ be a locally integrable function on $\mathbb{R}$ and $\nu \in \mathbb{R}$, $\nu > -1$, such that

\[
\lim_{x \to \pm \infty} \frac{F(x)}{|x|^{\nu} L(|x|)} = C_{\pm},
\]

where $L$ is slowly varying at $\infty$. Then $F \overset{L}{\sim} g$ at $\pm \infty$ related to $k^\nu L(k)$, where

\[
g(x) = C_+ x_+^\nu + C_- x_-^\nu.
\]

**Proof.** Let us put $F_+(x) = H(x)F(x)$ and $F_-(x) = H(-x)F(x)$, $x \in \mathbb{R}$, we can now apply Example 3 in 2.3 to each $F_\pm$ (see also Theorem 2.8). \quad \square
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We have the following structural theorem, it will be extended in 2.10.

**Theorem 2.14.** Let $f \in \mathcal{D}'(\mathbb{R})$ and $f \not\sim g$ at $\pm \infty$ related to $k^{\nu}L(k)$, where $g \neq 0$ and $\nu \in \mathbb{R} \setminus (-\mathbb{N})$. There are $m \in \mathbb{N}_0$ and a locally integrable function $F$ such that

$$f = F^{(m)} \quad \text{and} \quad \lim_{x \to \pm \infty} \frac{F(x)}{|x|^{\nu+m}L(|x|)} = C_{\pm},$$

where $(C_{+}, C_{-}) \neq (0, 0)$.

**Proof.** Since $f \in \mathcal{S}'(\mathbb{R})$ (Theorem 2.10), let $f = f_{+} + f_{-}$, where $f_{+} \in \mathcal{S}'_{+}$ and $f_{-} \in \mathcal{S}'_{-}$ (supp $f_{-} \subset (-\infty, 0]$). The Corollary of Theorem 2.10 implies that for every $\phi \in \mathcal{S}(\mathbb{R})$,

$$\left\langle \frac{f(kx)}{k^{\nu}L(k)}, \phi(x) \right\rangle \to \left\langle g(x), \phi(x) \right\rangle \quad \text{as} \quad k \to \infty.$$

As in the proof of the Corollary of Theorem 2.10, one may choose $f_{\pm}$ satisfying the additional requirement

$$\left\langle \frac{f_{\pm}(kx)}{k^{\nu}L(k)}, \phi(x) \right\rangle \to \left\langle \tilde{C}_{\pm} f_{\nu+1}(\pm x), \phi(x) \right\rangle \quad \text{as} \quad k \to \infty.$$

By Theorem 2.2, there exist locally integrable functions $F_{1}$ and $F_{2}$ with supp $F_{1} \subset [0, \infty)$, supp $F_{2} \subset (-\infty, 0]$, and $m \in \mathbb{N}_0$ such that

$$f_{+}(x) = F_{1}^{(m)}(x), \quad f_{-}(x) = F_{2}^{(m)}(x), \quad x \in \mathbb{R},$$

and

$$\lim_{x \to \infty} \frac{F_{1}(x)}{x^{\nu+m}L(x)} = C_{+}, \quad \lim_{x \to -\infty} \frac{F_{2}(x)}{|x|^{\nu+m}L(|x|)} = C_{-}.$$

This completes the proof. $\square$

The proof of the next result can be found in [111].

**Theorem 2.15.** Let $f \in \mathcal{S}'(\mathbb{R})$ and $\phi_{0} \in \mathcal{D}(\mathbb{R})$ such that $\int \phi_{0}(t)dt = 1$. Let $f' \not\sim g$ at $\pm \infty$ related to $k^{\nu}L(k)$, $v \in \mathbb{R}$, $g \neq 0$, and

$$\left\langle \frac{f(kx)}{k^{\nu+1}L(k)}, \phi_{0}(x) \right\rangle \to \left\langle g_{0}(x), \phi_{0}(x) \right\rangle,$$

where $g_{0} \in \mathcal{S}'(\mathbb{R})$ and $g_{0}' = g$. Then $f \not\sim g_{0}$ at $\pm \infty$ related to $k^{v+1}L(k)$.
Recall that the Fourier transform of a tempered distribution $f$ is denoted by $\mathcal{F}(f)$ or $\hat{f}$. We shall analyze in 2.8 the quasi-asymptotics of a distribution at zero (cf. Definition 2.5). In the following theorem, which is very useful, we already compare these two notions via the Fourier transform.

**Theorem 2.16.** Let $f \in \mathcal{D}'(\mathbb{R})$ and $v \in \mathbb{R} \setminus (-\mathbb{N})$. If

$$f \sim g \text{ at } \pm \infty \text{ related to } k^v L(k),$$

with $g \neq 0$, then

$$\lim_{k \to \infty} \frac{\hat{f}(x/k)}{(1/k)^{-v-1}L_1(1/k)} = \hat{g}(x) \text{ in } \mathcal{S}'(\mathbb{R}),$$

where $L_1(\cdot) = L(1/\cdot)$ is slowly varying at the origin.

Conversely, if $f \in \mathcal{S}'(\mathbb{R})$ and (2.30) holds with $v \in \mathbb{R}$, then (2.29) holds, as well.

**Proof.** Let $\phi \in \mathcal{S}(\mathbb{R})$. By the Corollary of Theorem 2.10, $f \in \mathcal{S}'(\mathbb{R})$ and

$$\left\langle \frac{f(kx)}{k^v L(k)} \hat{\phi}(x) \right\rangle = \left\langle \frac{\hat{f}(x/k)}{(1/k)^{-v-1}L_1(1/k)} \phi(x) \right\rangle, \quad k > 0.$$  

This implies the assertion. \hfill \Box

**Theorem 2.17.** Let $T \in \mathcal{E}'(\mathbb{R})$ and $T \sim g_1 \text{ at } \pm \infty \text{ related to } k^v$, $v \in -\mathbb{N}$, $g_1 \neq 0$. Let $f \in \mathcal{D}'(\mathbb{R})$ and $f \sim g \text{ at } \pm \infty \text{ related to } k^\alpha L(k)$, $\alpha \in \mathbb{R} \setminus (-\mathbb{N})$, $g \neq 0$. Then $T * f \sim g_1 * g \text{ at } \pm \infty \text{ related to } k^{\alpha+v+1}L(k)$.

**Proof.** Let $\phi \in \mathcal{S}(\mathbb{R})$. Using the properties of the Fourier transform, we have

$$\left\langle \frac{(T * f)(kx)}{k^{\alpha+v+1}L(k)} \hat{\phi}(x) \right\rangle = \left\langle \frac{\hat{T}(x/k) \hat{f}(x/k)}{k^\alpha \cdot (1/k)^{-v-1}L(1/k)} \phi(x) \right\rangle = \left\langle \frac{\hat{f}(x/k)}{(1/k)^{-v-1}L(1/k)} \frac{\hat{T}(x/k)}{(1/k)^{-\alpha-1}L(1/k)} \phi(x) \right\rangle.$$  

(2.31)

Since $\hat{T}$ is an entire function of polynomial growth when $|x| \to \infty$, it must be of the form $\hat{T}(x) = x^{-v-1}T_1(x)$, $x \in \mathbb{R}$, where $T_1$ is an entire function of polynomial growth such that $T_1(0) = C \neq 0$. All the derivatives of $\hat{T}$ are of polynomial growth when $|x| \to \infty$. So, the same holds for $T_1$. This implies that for any $\phi \in \mathcal{S}(\mathbb{R})$

$$\frac{1}{k^{v+1}} (x/k)^{-v-1} T_1(x/k) \phi(x) = x^{-v-1} T_1(x/k) \phi(x) \to x^{-v-1} T_1(0) \phi(x),$$
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$k \to \infty$, in the sense of convergence in $\mathcal{S}(\mathbb{R})$. Let us note that $\hat{g}_1(x) = x^{-v-1}T_1(0), x \in \mathbb{R}$.

In the spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ the strong and weak sequential convergence are equivalent. This implies that

$$\langle \hat{f}(x/k), \hat{T}(x/k) \rangle \to \langle \hat{g}(x), \hat{g}_1(x) \phi(x) \rangle = \langle (g_1 * g)(x), \phi(x) \rangle \kappa \to \infty.$$

By (2.31), the claim follows from Theorem 2.16. \hfill \Box

**Theorem 2.18.** Let $f \in \mathcal{D}'(\mathbb{R})$ and $\{f(kx)/k^\alpha L(k); k > a\}$, $\alpha \in \mathbb{R}\setminus(-\mathbb{N})$, be a bounded subset of $\mathcal{D}'(\mathbb{R})$. Let $T \in \mathcal{E}'(\mathbb{R})$ and $T \sim g_1$ at $\pm \infty$ related to $k^{-1}, g_1 \neq 0$. If $T * f \sim g_2$ at $\pm \infty$ related to $k^\alpha L(k), g_2 \neq 0$, then $f \sim g$ at $\pm \infty$ related to $k^\alpha L(k)$ and $g_1 * g = g_2$. (Note, $g_1 \equiv C\delta$).

**Proof.** The same arguments, as in the proof of Theorem 2.10 yield that $f \in \mathcal{S}'(\mathbb{R})$ and that

$$\{f(kx)/k^\alpha L(k); k > a\}$$

is a bounded subset of $\mathcal{S}(\mathbb{R})$. With the same arguments as above, we have ($\phi \in \mathcal{S}(\mathbb{R})$)

$$\left\langle \frac{f(x/k)}{(1/k)^{-\alpha-1}L_1(1/k)}, \phi(x) \right\rangle \to 0 \quad \text{as} \quad k \to \infty.$$

This implies the assertion. \hfill \Box

2.8 Quasi-asymptotics at the origin

**Definition 2.5.** Let $f \in \mathcal{F}'(\mathbb{R})$ and $c(x), x \in (0, a), a > 0$, be a positive measurable function. It is said that $f$ has quasi-asymptotics at 0 in $\mathcal{F}'(\mathbb{R})$ related to $c(1/k)$ if there is $g \in \mathcal{F}'(\mathbb{R})$ such that

$$\lim_{k \to \infty} \left\langle \frac{f(x/k)}{c(1/k)}, \varphi(x) \right\rangle = \langle g(x), \varphi(x) \rangle, \varphi \in \mathcal{F}(\mathbb{R}).$$

We write in short: $f \sim g$ at 0 related to $c(1/k)$ in $\mathcal{F}'(\mathbb{R})$. 

Remark. The limit in Definition 2.5 may be formulated as

\[ \lim_{\varepsilon \to +0} \left\langle \frac{f(\varepsilon x)}{c(\varepsilon)}, \phi(x) \right\rangle = \left\langle g(x), \phi(x) \right\rangle, \quad \phi \in \mathcal{F}(\mathbb{R}). \]

Therefore, we often denote also quasi-asymptotics at 0 by

\[ f(\varepsilon x) \sim c(\varepsilon)g(x), \quad \varepsilon \to 0^+ \text{ in } \mathcal{F}'(\mathbb{R}), \]

or simply by \( f \sim g \) at 0 related to \( c(\varepsilon) \).

If \( c = 1 \) then this Definition 2.5 is a slight generalization of the well-known Lojasiewicz definition of a “value at 0” of a distribution (see [93] and [105]), and it leads to a notion of jump behavior for distributions [173], [176], [178], [181] and [57]. Note that in the Lojasiewicz definition \( \varepsilon \to 0 \) from both sides.

We list some properties of the quasi-asymptotics at 0 for distributions, we omit the proofs if they are similar to those of the corresponding properties of the quasi-asymptotics at infinity. For details see [112].

The next proposition shows that we may always assume \( c \) is regularly varying at the origin (cf. 0.3). The proof is essentially the same as that of (i) in Theorem 2.10.

**Proposition 2.7.** Let \( f \in \mathcal{D}'(\mathbb{R}) \) and \( c \) satisfy the conditions of Definition 2.5. Assume further that \( g \neq 0 \). Then, for some real number \( v \) and some slowly varying function \( L \) at 0+

\[ c(x) = x^v L(x), \quad x \in (0, a). \]

Moreover, \( g \) is homogeneous with the degree of homogeneity \( v \).

Some obvious properties of the quasi-asymptotics at 0 in \( \mathcal{S}'(\mathbb{R}) \) are given in the following proposition.

**Proposition 2.8.** Let \( f \in \mathcal{S}'(\mathbb{R}) \) and \( f \sim g \) at 0 related to \( \varepsilon^v L(\varepsilon) \) in \( \mathcal{S}'(\mathbb{R}) \). Then:

(i) \( f^{(m)} \sim g^{(m)} \) at 0 related to \( \varepsilon^{v-m} L(\varepsilon) \) in \( \mathcal{S}'(\mathbb{R}) \), \( m \in \mathbb{N} \);
(ii) \( x^m f(x) \sim x^m g(x) \) at 0 related to \( \varepsilon^{v+m} L(\varepsilon) \) in \( \mathcal{S}'(\mathbb{R}) \), \( m \in \mathbb{N} \).

The same assertions hold for the quasi-asymptotics at 0 in \( \mathcal{D}'(\mathbb{R}) \).
We have seen (see remark after Proposition 2.4) that the quasi-asymptotics at infinity is not in general a local property. The next proposition asserts that the quasi-asymptotics at 0 is a local property.

**Proposition 2.9.** Let \( f \in \mathcal{D}'(\mathbb{R}) \) and \( f \not\sim g \) at 0 related to \( \epsilon^\alpha L(\epsilon) \) in \( \mathcal{D}'(\mathbb{R}) \), and let \( f_1 \in \mathcal{D}'(\mathbb{R}) \) be such that \( f = f_1 \) in some neighborhood of zero. Then \( f_1 \not\sim g \) at 0 related to \( \epsilon^\alpha L(\epsilon) \), as well.

**Proof.** The assertion follows from

\[
\langle f(x), \varphi(x/\epsilon) \rangle = \langle f_1(x), \varphi(x/\epsilon) \rangle
\]

which holds for any \( \varphi \in \mathcal{D}(\mathbb{R}) \) if \( \epsilon < \epsilon_0(\varphi) \).

**Remark.** The same assertion holds for the quasi-asymptotics at 0 in \( \mathcal{S}'(\mathbb{R}) \). This was proved in [34], Lemma 1.6. This claim also follows directly if we combine Proposition 2.9 with Theorem 2.35 from 2.11.1.

**Proposition 2.10.** (Theorem 3, Chapter I, 3.3 in [192]) Let \( f \in \mathcal{S}'(\mathbb{R}) \) be a locally integrable function in \((-a, a), a > 0 \). Let \( c(\epsilon) = \epsilon^\alpha L(\epsilon), \alpha > -1 \), where as usual \( L \) is slowly varying at 0+. If

\[
\lim_{x \to \pm 0} f(x)/c(|x|) = C_\pm,
\]

then \( f \) has the quasi-asymptotics at zero in \( \mathcal{S}'(\mathbb{R}) \) related to \( c \) and

\[
\lim_{\epsilon \to 0^+} f(\epsilon x)/c(\epsilon) = C_+ x_+^\alpha + C_- x_-^\alpha \text{ in } \mathcal{S}'(\mathbb{R}).
\]

**Proof.** According to the remark after Proposition 2.9, it is sufficient to prove that for any \( \varphi \in \mathcal{S}(\mathbb{R}) \) the limits of the following expressions exist, when \( \epsilon \to 0^+ \),

\[
\int_0^a f(\epsilon x)/c(\epsilon) \varphi(x)dx \quad \text{and} \quad \int_{-a}^0 f(\epsilon x)/c(\epsilon) \varphi(x)dx.
\]

Let us consider the first one:

\[
\lim_{k \to \infty} \int_0^a f(\epsilon x)/c(\epsilon) \varphi(x)dx = \lim_{k \to \infty} \int_0^a \frac{c(\epsilon x)}{c(\epsilon)} \frac{f(\epsilon x)}{c(\epsilon)} \varphi(x)dx.
\]

But since the term in the second integral is dominated by an \( L^1 \) function \( (c(\epsilon x)/c(\epsilon) = O(\epsilon^{\alpha-\sigma}) \), where \( \alpha > \sigma - 1 \), \( [148] \)), it is possible to pass the limit under the integral. The second expression can be considered in the same way.
Proposition 2.10 provides a relation between the asymptotic behavior at zero of a locally integrable function and the quasi-asymptotics at zero of the distribution defined by it.

The next Proposition is a direct consequence of Theorem 2.15 and Theorem 2.16.

**Proposition 2.11.** Let \( f \in S'(\mathbb{R}) \) such that \( xf^{q} \sim g \) at 0 related to \( \varepsilon v L(\varepsilon) \), \( v \in \mathbb{R} \setminus \{-N\} \) in \( S'(\mathbb{R}) \). Let \( \varphi_0 \in D(\mathbb{R}) \) such that \( \int \varphi_0(x) dx = 1 \) and

\[
\left\langle \frac{f(\varepsilon x)}{\varepsilon^v L(\varepsilon)}, \varphi_0(x) \right\rangle \to \left\langle g_0(x), \varphi_0(x) \right\rangle \quad \text{as} \quad \varepsilon \to 0^+
\]

such that \( g_0 \in S'(\mathbb{R}) \) and \( xg_0(x) = g(x), \; x \in \mathbb{R} \). Then, \( f \sim g_0 \) at 0 related to \( \varepsilon v L(\varepsilon) \) in \( S'(\mathbb{R}) \) (\( g \) and \( g_0 \) are homogeneous of order \( v + 1 \) and \( v \), respectively).

Proposition 2.10 and (i) of Proposition 2.8 directly yield

**Proposition 2.12.** Let \( f \in S'(\mathbb{R}) \) and \( f = F^{(m)} \) in some neighborhood of \( 0 \), where \( m \in N_0 \) and \( F \) is a locally integrable function such that for some \( v > -1 \) and some slowly varying function \( L \),

\[
\lim_{x \to \pm 0} \frac{F(x)}{|x|^v L(|x|)} = C_{\pm}.
\]

Then \( f \sim g \) at 0 related to \( \varepsilon^v L(\varepsilon) \) in \( S'(\mathbb{R}) \), where \( g = (C_+ x_+^v + C_- x_-^v)^{(m)} \).

We now discuss an structural theorem. A more complete result will be the subject of 2.10.3–2.10.5.

**Theorem 2.19.** Let \( f \in D'(\mathbb{R}) \) have quasi-asymptotics at 0 in \( D'(\mathbb{R}) \) related to \( \varepsilon^v L(\varepsilon) \). If \( v > 0 \) or if \( v > -1 \) but \( L \) is bounded on some interval \((0, a), \; a > 0\), then there exist a continuous function \( F \), defined on \((-1, 1)\), an integer \( m \), and constants \( C_+, C_- \), such that

\[
f = F^{(m)} \quad \text{and} \quad \lim_{x \to \pm 0} \frac{F(x)}{|x|^{v+m} L(|x|)} = C_{\pm}. \quad (2.32)
\]

**Proof.** The proof of this theorem is similar to the proof of the Lojasiewicz structural theorem for a distribution having a value at 0, given
in [102], pp. 49–52. However, it is necessary to make several non-trivial refinements in the quoted argument.

Let \( I = (-2, 2) \). Since \( f(\varepsilon x)/L(\varepsilon) \to g(x) \) in \( \mathcal{D}'(\mathbb{R}) \), \( \varepsilon \to 0^+ \), there is a family of continuous functions \( F_\varepsilon, \varepsilon \in (0, \varepsilon_0] \), defined on \( I \), an \( m \in \mathbb{R}_0 \) such that

\[
\frac{f(\varepsilon x)}{\varepsilon^m L(\varepsilon)}, \quad x \in I, \ \varepsilon \in (0, \varepsilon_0] \]

and \( F_\varepsilon(x) \to g_1(x) \) uniformly on \( I \) when \( \varepsilon \to 0^+ \), where

\[
g_1(x) = \frac{(C_+ x^{v+m} + C_- x^{v-m})/m!}{L(\varepsilon)}. \]

With no loss of generality, we assume that \( \varepsilon_0 \geq 1 \). Let us put

\[
\tilde{F}_\varepsilon(x) = F_\varepsilon(x)\varepsilon^m L(\varepsilon), \quad x \in I, \ 0 < \varepsilon \leq 1.
\]

From \( \tilde{F}_\varepsilon(x) = f \) it follows \( (\varepsilon^{-m} F_\varepsilon(\varepsilon x))^{(m)} = f(\varepsilon x), \ x \in I, \ \varepsilon \in (0, 1] \). So, because \( \tilde{F}_\varepsilon^{(m)}(x) = f(\varepsilon x) \), we obtain

\[
(\tilde{F}_\varepsilon(\varepsilon x) - \varepsilon^m \tilde{F}_\varepsilon(x))^{(m)} = 0, \quad x \in I, \ \varepsilon \in (0, 1].
\]

This implies that there is a polynomial which depends on \( \varepsilon \) such that

\[
\tilde{F}_\varepsilon(x) = \varepsilon^{-m} (\tilde{F}_\varepsilon(\varepsilon x) + b_0(\varepsilon) + b_1(\varepsilon)\varepsilon x + \cdots + b_{m-1}(\varepsilon)(\varepsilon x)^{m-1}),
\]

\( x \in I, \ \varepsilon \in (0, 1] \). We have

\[
\frac{\tilde{F}_\varepsilon(x)}{\varepsilon^m L(\varepsilon)} - g_1(x) = \frac{1}{\varepsilon^{m+v} L(\varepsilon)} \left[ \tilde{F}_\varepsilon(\varepsilon x) - g_1(\varepsilon x) L(\varepsilon|x|) \right]
\]

\[
+ b_0(\varepsilon) + \cdots + b_{m-1}(\varepsilon)(\varepsilon x)^{m-1}
\]

\[
+ g_1(\varepsilon) \left( \frac{L(\varepsilon|x|)}{L(\varepsilon)} - 1 \right), \quad x \in I, \ \varepsilon \in (0, 1].
\]

We obtain that for any \( x \in I \setminus \{0\} \)

\[
\frac{1}{\varepsilon^{m+v} L(\varepsilon)} (G(\varepsilon x) + b_0(\varepsilon)\varepsilon x + \cdots + b_{m-1}(\varepsilon)(\varepsilon x)^{m-1}) \to 0
\]

as \( \varepsilon \to 0^+ \), where we put \( G(x) = \tilde{F}_\varepsilon(x) - g_1(x)L(|x|) \).

Note that the last limit is not uniform in general because we have only the following property of a slowly varying function:

\[
\frac{L(\varepsilon|x|)}{L(\varepsilon)} \to 1, \ \varepsilon \to 0^+, \quad \text{uniformly for} \quad |x| \in [a, b], \ b > a > 0.
\]
Let us fix $m$ points $x_1, \ldots, x_m \in I$ such that $x_i \neq 0$, $i = 1, \ldots, m$, and let
\[ d = \frac{1}{2} \min\{|x_i|; i = 1, \ldots, m\}, \quad J = \{x; |x| > d\} \cap I. \]
Because of the quoted property of $L$, we have
\[ \frac{1}{\varepsilon^{m+v}L(\varepsilon)}(G(\varepsilon x) + b_0(\varepsilon) + b_1(\varepsilon)x + \cdots + b_{m-1}(\varepsilon)(\varepsilon x)^{m-1}) \to 0, \quad \varepsilon \to 0^+, \]
uniformly on $J$. This implies that for some monotonously increasing function $\eta(\varepsilon)$, $\varepsilon > 0$, $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0^+$, there holds
\[ |G(\varepsilon x) + b_0(\varepsilon) + b_1(\varepsilon)x + \cdots + b_{m-1}(\varepsilon)(\varepsilon x)^{m-1}| < \varepsilon^{m+v}L(\varepsilon)\eta(\varepsilon), \]
\[ x \in J, \quad \varepsilon \in (0, 1]. \] (2.33)

Let $\beta > 0$ and $\varepsilon < \beta \leq 2\varepsilon$. If we put in (2.33) $\beta$ instead of $\varepsilon$ and $x_i\varepsilon/\beta$ instead of $x$, $i = 1, \ldots, m$, (note $x_i\varepsilon/\beta \in J$, $i = 1, \ldots, m$), we obtain
\[ |G(x_i) + b_0(\beta) + \varepsilon b_1(\beta)x_i + \cdots + \varepsilon^{m-1}b_{m-1}(\beta)x_i^{m-1}| < \beta^{m+v}L(\beta)\eta(\beta). \]
From (2.33) and the last inequality it follows
\[ |b_0(\varepsilon) - b_0(\beta) + \varepsilon b_1(\beta)x_i + \cdots + \varepsilon^{m-1}(b_{m-1}(\varepsilon) - b_{m-1}(\beta))x_i^{m-1}| < 2\beta^{m+v}L(\beta)\eta(\beta). \]

Now, in the same way as in [102], p. 51 one can prove
\[ |b_i(\varepsilon) - b_i(\beta)| < 2^{i+1}K\eta(\beta)L(\beta)\beta^{m+i-1}, \]
\[ \varepsilon < \beta \leq 2\varepsilon, i = 0, \ldots, m - 1, \] (2.34)
where $K$ is a suitable constant. Let $\varepsilon < \beta < 1/2$. Take $r \in \mathbb{N}_0$ such that $\beta/2^{r+1} < \varepsilon < \beta/2^r$. (2.34) implies
\[ |b_i(\beta/2^r) - b_i(\beta/2^j)| < 2^{j+1}K\eta(\beta/2^{j-1})L(\beta/2^{j-1})(\beta/2^{j-1})^{m+i-1}, \] (2.35)
\[ i = 0, \ldots, m - 1; \quad j = 1, \ldots, r, \quad \text{and} \]
\[ |b_i(\beta/2^r) - b_i(\varepsilon)| < 2^{i+1}K\eta(\beta/2^r)L(\beta/2^r)(\beta/2^r)^{m+i-1}, \] (2.36)
\[ i = 0, \ldots, m - 1. \]

Let $v > 0$ and $C = \sup\{tL(t); t \in (0, \beta)\}$. (2.35) and (2.36) imply ($0 < \varepsilon < \beta < 1/2$)
\[ |b_i(\varepsilon) - b_i(\beta)| \leq 2^{i+1}K\eta(\beta)C \left( \sum_{j=1}^{r+1} (\beta/2^j)^{-1} \right)^{m+i-1} \]
\[ \leq 2^{i+v+m+2}K\eta(\beta)C\beta^{m+i-1}, \quad i = 0, 1, \ldots, m - 1. \] (2.37)
Note that the assumption \( v > 0 \) is essential in the above inequality. From (2.37) it follows
\[
|b_i - b_i(\epsilon)| < \frac{1}{\epsilon} K \eta(\epsilon) \varepsilon^{m+v-i-1}, \quad i = 0, \ldots, m-1,
\]
where \( K \) is a suitable constant.

From (2.33) and the last inequality it follows
\[
|G(\varepsilon x) + b_0 + \varepsilon b_1 x + \cdots + \varepsilon^{m-1} b_{m-1} x^{m-1}| < K_1 \varepsilon^{m+v} L(\varepsilon) \eta(\varepsilon), \quad (2.38)
\]
x \in J, \quad 0 < \varepsilon \leq 1 \text{ (with suitable } K_1).\]

Let us show that the function
\[
F(x) = G(x) + b_0 + b_1 x + \cdots + b_{m-1} x^{m-1} + g_1(x) L(|x|), \quad x \in (-2, 2),
\]
satisfies the conditions of the Theorem. Clearly, \( F(m) = f. \) Put in (2.38)
x = \pm 1. We obtain
\[
|F(\pm \varepsilon) - g(\pm \varepsilon) L(\varepsilon)| < K_1 \varepsilon^{m+v} L(\varepsilon) \eta(\varepsilon), \quad 0 < \varepsilon \leq 1.
\]

Let \( v > -1 \) and \( L(x) < C, \quad x \in (0, a). \) Then from (2.35) and (2.36) it follows (for \( 0 < \varepsilon < \beta, \quad \beta < a, \quad \beta < 1/2)\)
\[
|b_i(\varepsilon) - b_i(\beta)| \leq 2^{i+1} K C \eta(\beta) \sum_{j=1}^{r+1} \left( \frac{\beta}{2^{j-1}} \right) m+v-i
\]
\[
\leq 2^{i+1} K C \eta(\beta) \beta^{m+v-i} \sum_{j=1}^{r+1} (1/2^{j-1}) m+v-i, \quad i = 0, \ldots, m-1.
\]

Now, the proof follows as in the previous case. The proof is complete. \( \square \)

The assertion of Theorem 2.19 also holds for \( v < 0. \) This situation is analyzed in the following theorem. The proof of it has been given in [116]. We omit the proof because more general results will be given in 2.10.3.

**Theorem 2.20.** Let \( f \in S'(\mathbb{R}) \) and let \( f \) have the quasi-asymptotic behavior at 0 in \( S'(\mathbb{R}) \) relate to \( \alpha^v L(\alpha), \) where \( v \in (-\infty, 0), \quad v \neq -1, -2, \ldots \) and \( L \) is bounded on some interval \((0, a), \ a > 0. \) Then there exist a continuous function \( F \) defined on \((-1, 1), \) an integer \( m \) and \((C_+, C_-) \neq (0, 0) \) such that (2.32) holds.
Remarks. 1. If \( v = 0 \) and \( L \equiv 1 \), then Theorem 2.19 generalizes the well-known Lojasiewicz structural theorem ([93]) for a distribution which has a value at 0.

2. Let \( f \in S'(\mathbb{R}) \) and \( f \overset{g}{\sim} g \) at 0 related to \( c(\varepsilon) \) in \( \mathcal{D}'(\mathbb{R}) \). The question is: Does the same hold in \( S'(\mathbb{R}) \)? We shall prove in this section that for \( c(\varepsilon) = \varepsilon^v L(\varepsilon), \varepsilon < a \), the answer to the question is affirmative if \( v > 0 \) or if \( 0 \geq v > -1 \) and \( L \) is bounded in some interval \( (0, \mu) \), \( \mu > 0 \). Otherwise, this question remained open for quite long time. A complete affirmative answer has been recently obtained in [186], it will be presented below in 2.11 (Theorem 2.35).

3. Theorems 2.19 and 2.20 describes the structure of quasi-asymptotics only under restrictions over \( \nu \) and \( L \). The complete structural theorems will be discussed in 2.10.

Proposition 2.12 and Theorem 2.19 directly imply

**Theorem 2.21.** Let \( f \in S'(\mathbb{R}) \) satisfy conditions of Theorem 2.19. Then \( f \overset{g}{\sim} g \) at 0 related to \( \varepsilon^v L(\varepsilon) \) in \( S'(\mathbb{R}) \).

Let us denote by \( Z \) the space of Fourier transforms of elements from \( \mathcal{D}(\mathbb{R}) \) supplied by the convergence structure transferred from \( \mathcal{D}(\mathbb{R}),(Z = F(\mathcal{D})) \). Let \( f \in S'(\mathbb{R}) \). As usual, we write \( f \overset{g}{\sim} g \) at 0 related to \( \varepsilon^v L(\varepsilon) \) in \( Z' \) if \( g \in Z' \), and

\[
\lim_{\varepsilon \to 0^+} \left\langle \frac{f(\varepsilon x)}{\varepsilon^v L(\varepsilon)}, \phi(x) \right\rangle = \left\langle g(x), \phi(x) \right\rangle, \quad \phi \in Z.
\]

Using the Fourier transform and Theorem 2.10 (ii) one can easily obtain that \( g \in S'(\mathbb{R}) \).

For \( v < 0 \), we have the ensuing related result. above ([112]).

**Theorem 2.22.** Let \( f \in S'(\mathbb{R}) \) and \( f \overset{g}{\sim} g \) at 0 related to \( \varepsilon^v L(\varepsilon) \) in \( Z' \), where \( v < 0 \), \( v \notin -\mathbb{N} \), and \( g \neq 0 \). Then \( f \overset{g}{\sim} g \) at 0 related to \( \varepsilon^v L(\varepsilon) \) in \( S'(\mathbb{R}) \).

**Proof.** Apply Fourier transform, (iii) in Theorem 2.10, and then Theorem 2.16.
2.9 Quasi-asymptotic expansions

Chapter III, §10 in [192] and §4 in [32] were devoted to the quasi-asymptotic expansion of tempered distributions with support in \([0, \infty)\). Two kind of expansions were defined therein. We will recall these definitions but in \(\mathcal{F}'_+\). We also survey other definitions appearing in the literature.

Let \(\{f_\alpha; \alpha \in \mathbb{R}\}\) be the family of tempered distributions belonging to \(\mathcal{S}'_+\) which is defined in 0.4.

We denote by \(\Sigma_\infty\) (by \(\Sigma_0\)) the set of all positive slowly varying functions at \(\infty\) (at \(0^+\)).

Let \(\alpha \in \mathbb{R}\) and \(L \in \Sigma_\infty\) (\(L \in \Sigma_0\)). We introduce another family of tempered distributions:

\[
(f_L)_{\alpha+1}(t) = \begin{cases} 
H(t)L(t)t^\alpha/\Gamma(\alpha+1), & \alpha > -1, \\
D^\alpha(f_L)_{\alpha+n+1}(t), & \alpha \leq -1, \alpha + n > -1,
\end{cases} \tag{2.39}
\]

where \(n\) is the smallest natural number such that \(\alpha + n > -1\).

Obviously, \((f_L)_{\alpha+1} \sim f_{\alpha+1}\) at \(\infty\) (at \(0^+\)) related to \(k^\alpha L(k)\) (to \((1/k)^\alpha L(1/k))\).

**Definition 2.6.** ([192]) A distribution \(g \in \mathcal{S}'_+\) is said to have a closed quasi-asymptotic expansion of order \(\alpha\) and of length \(\ell, 0 \leq \ell \leq \infty\) if there exist \(N \in \mathbb{N}, \alpha_j \in \mathbb{R}\) and \(c_j \in \mathbb{C}, j = 1, \ldots, N\), such that

\[
\lim_{k \to \infty} \frac{1}{k^{\alpha+\ell}} \left( g(kt) - \sum_{j=1}^{N} c_j f_{\alpha_j+1}(kt) \right) = 0, \quad \text{in } \mathcal{S}_+.\]

**Definition 2.7.** ([192]) A distribution \(g \in \mathcal{S}'_+\) is said to have an open quasi-asymptotic expansion of order \(\alpha\) and of length \(\ell, 0 < \ell \leq \infty\) if for every \(\ell_1 < \ell\), \(g\) has closed quasi-asymptotic expansion of order \(\alpha\) and of length \(\ell_1\).

In [101, 117, 122] these two definitions were slightly altered and extended by using the family \(\{(f_L)_{\alpha}; \alpha \in \mathbb{R}\}\) instead of the family \(\{f_\alpha; \alpha \in \mathbb{R}\}\) (cf. (2.39)). In the next definition we assume that \(\{(f_L)_{\alpha}; \alpha \in \mathbb{R}\} \subset \mathcal{F}'_+\).

**Definition 2.8.** We say that an \(f \in \mathcal{F}'_+\) has a closed quasi-asymptotic expansion at \(\infty\) (at \(0^+\)) of order \((\alpha, L) \in \mathbb{R} \times \Sigma_\infty\) (of order \((\alpha, L) \in \)
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$R \times \Sigma_0$ and of length $\ell$, $0 \leq \ell < \infty$, related to $k^{a-\ell} L_0(k)$ (related to $(1/k)^{a+\ell} L_0(1/k)$) if $f$ has quasi-asymptotics at $\infty$ (at $0^+$) related to $k^a L(k)$ ($(1/k)^a L(1/k)$) and if there exist $\alpha_i \in R$, $L_i \in \Sigma_\infty(L_i \in \Sigma_0)$, and $c_i \in C$, $i = 1, \ldots, N$, $N \in N$, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_N$ ($\alpha_1 \leq \alpha_2 \cdots \leq \alpha_N$), and such that $f$ is of the form

$$f(t) = \sum_{i=1}^N c_i (f_{L_i})_{\alpha_{i+1}}(t) + h(t),$$

where for every $\phi \in F(R)$

$$\lim_{k \to \infty} \left< \frac{h(kt)}{k^{a-\ell} L_0(k)}, \phi(t) \right> = 0, \quad \left( \lim_{k \to \infty} \left< \frac{h(t/k)}{(1/k)^{a+\ell} L_0(1/k)}, \phi(t) \right> = 0 \right).$$

We write in short:

$$f \overset{q.e.}{\sim} \sum_{i=1}^N c_i (f_{L_i})_{\alpha_{i+1}} \text{ at } \infty \text{ (at } 0^+) \text{ of order } (\alpha, L).$$

Obviously, we shall assume that $c_i \neq 0$ and that $\alpha_N \geq \alpha - \ell$ ($\alpha_N \leq \alpha + \ell$).

We shall always assume that in representation (2.40), $\alpha_1 > \alpha_2 > \cdots > \alpha_N$ ($\alpha_1 < \alpha_2 < \cdots < \alpha_N$), because $(f_{L_j})_{\beta_{j+1}} + (f_{L_k})_{\beta_{j+1}} \sim (f_{L_j+L_k})_{\beta_{j+1}}$.

Observe $(f_{L_j})_{\beta_{j+1}}$ and $(f_{L_k})_{\beta_{j+1}+1}$ have the same quasi-asymptotics at $\infty (0^+)$ if and only if $\beta_1 = \beta_2$ and $L_j \sim L_k$. So, we have:

**Proposition 2.13.** Let $f \in F'_+$ satisfy the conditions of Definition 2.8 and assume that there are two representations of $f$, with the same length $l$,

$$f(t) = \sum_{i=1}^N c_i (f_{L_i})_{\alpha_{i+1}} + h(t), \quad f(t) = \sum_{i=1}^M \tilde{c}_i (f_{L_i})_{\tilde{\alpha}_{i+1}} + \tilde{h}(t)$$

for which all the assumptions given above hold. Then, $M = N$, $\alpha_1 = \tilde{\alpha}_1, \ldots, \alpha_N = \tilde{\alpha}_N$, $L_1 \sim \tilde{L}_1, \ldots, L_N \sim \tilde{L}_N$.

**Examples:** All examples are in $S'_+(R)$.

1. We have that $\sum_{r=1}^\infty \frac{H(x-1)}{r! x^r}$, $x \in R$, converges uniformly to $H(x - 1)e^{1/x}$ but

$$H(x - 1)e^{1/x} \overset{q.e.}{\sim} H(x) + ((\log x)_+)' \text{ at } \infty \text{ of order } (0, L \equiv 1)$$
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related to $k^{-1} \log k$ and

$$H(x-1)e^{1/x} \overset{q.e.}{\sim} H(x) + ((\log x) + ') + \left(-1 + \sum_{r=2}^{\infty} \frac{1}{r!} \frac{1}{r-1}\right) \delta(x) \text{ at } \infty$$
of order $(0, L \equiv 1)$ related to $k^{-1}$.

2. $H(t-1)/t^{\beta} \overset{q.e.}{\sim} (\delta - \delta')/2$ at $\infty$ of order $(-1, L \equiv 1)$ related to $k^{-3} \log k$. Moreover, let $n > 2$; then for $j \leq n - 2$

$$H(t-1)/t^{\beta} \overset{q.e.}{\sim} \frac{1}{(n-1)!} \delta + \frac{1}{(n-2)!} \delta' + \cdots + \frac{(-1)^{n-2}}{n!} \delta^{(n-2)}$$
at $\infty$ of order $(-1, L \equiv 1)$ related to $k^{-\beta}$;

$$H(t-1)/t^{\beta} \overset{q.e.}{\sim} \frac{1}{(n-1)!} \delta + \frac{1}{(n-2)!} \delta' + \cdots + \frac{(-1)^{n-2}}{n!} \delta^{(n-2)}$$
at $\infty$ of order $(-1, L \equiv 1)$ related to $k^{-n} \log k$.

Following [32], we define the extended open quasi-asymptotic expansion.

**Definition 2.9.** An $f \in \mathcal{F}^*_+$ has open quasi-asymptotic expansion at $\infty$ (at $0^+$) of order $(\alpha, L) \in \mathbb{R} \times \Sigma_{\infty} ((\alpha, L) \in \mathbb{R} \times \Sigma_0)$ and of the length $s$, $0 < s \leq \infty$, if and only if for every $\ell$, $0 \leq \ell < s$, $f$ has closed quasi-asymptotic expansion of order $(\alpha, L)$ and of length $\ell$, related to $k^{\alpha - \ell} L_\ell(k)$ ($((1/k)^{\alpha + \ell} L_\ell(1/k))$).

By the same arguments as in Proposition 2.13 one can prove the following proposition:

**Proposition 2.14.** Let $f \in \mathcal{F}^*_+$ have open quasi-asymptotic expansion at $\infty$ (at $0^+$) of order $(\alpha, L)$ and of length $s$ and let $0 \leq \ell_1 < \ell_2 < s$. Suppose that

$$f \overset{q.e.}{\sim} \sum_{i=1}^{N} a_i (f_{\ell_i}(1)\alpha_{i+1} \at \infty \text{ (at } 0^+)$$
related to $k^{\alpha - \ell_1} L_{\ell_1}(k)((1/k)^{\alpha + \ell_1} L_{\ell_1}(1/k))$, and

$$f \overset{q.e.}{\sim} \sum_{i=1}^{M} b_i (f_{\ell_i}(1)\beta_{i+1} \at \infty \text{ (at } 0^+)$$
related to $k^{\alpha - \ell_2} L_{\ell_2}(k)((1/k)^{\alpha + \ell_2} L_{\ell_2}(1/k))$.

Then, $M \geq N$ and $a_i = b_i$, $\alpha_i = \beta_i$, $L_i \sim \tilde{L}_i$, $i = 1, \ldots, N.$
Let us note that if \( f \) has the closed quasi-asymptotic expansion at \( \infty \) of order \((\alpha, L)\) related to \( k^{\alpha}L(k) \), then for any \( s \leq \ell \), \( f \) has the open quasi-asymptotic expansion at \( \infty \) of order \((\alpha, L)\) and of length \( s \). A similar conclusion holds for the point \( 0^+ \) as well.

We will actually use the notation

\[
f q.e. \sim \sum_{i=1}^{N} c_i (f_{L_i})_{\alpha_i+1} [k^{\alpha_i}L_i(\varepsilon)] \quad (\varepsilon^{k_i}L_i(\varepsilon))
\]

and instead of the “open or closed quasi-asymptotic expansion of order \((\alpha, L)\) and length \( s \)”, we will just say that \( f \) has the quasi-asymptotic expansion at \( \infty \) or with respect to the given scale.

Let us redefine the notion of quasi-asymptotic expansion. We state our definition at \( 0^+ \), but one can do the same for the quasi-asymptotic expansion at \( \infty \).

We denote by \( \Lambda \) the set \( \mathbb{N} \) or a finite set of the form \( \{1, 2, \ldots, N\}, N \in \mathbb{N} \). In the second case, we shall sometimes use the symbol \( \Lambda_N \). In the following definition, for \( \alpha_i \in \mathbb{R} \) and \( L_i \in \Sigma_0 \), \( i \in \Lambda \), we assume that if \( i, j \in \Lambda \), \( i < j \), then \( \alpha_i \leq \alpha_j \) and if \( \alpha_i = \alpha_j \), then \( L_j(\varepsilon)/L_i(\varepsilon) \to 0, \varepsilon \to 0^+ \).

**Definition 2.10.** An \( f \in F'(\mathbb{R}) \) has quasi-asymptotic expansion at \( 0^+ \) related to \( \{\varepsilon^{\alpha_k}L_k(\varepsilon); k \in \Lambda\} \) if there are complex numbers \( A_k \neq 0, k \in \Lambda \), so that for any \( m \in \Lambda \)

\[
\text{w.lim}_{\varepsilon \to 0^+} (f - \sum_{k=1}^{m} A_k (f_{L_k})_{\alpha_k+1} (\varepsilon x)) \varepsilon^{\alpha_m}L_m(\varepsilon) = 0 \text{ in } F'(\mathbb{R}).
\]

We write in short:

\[
f q.e. \sum_{k \in \Lambda} A_k (f_{L_k})_{\alpha_k+1} [\varepsilon^{\alpha_k}L_k(\varepsilon)],
\]

or simply \( f q.e. \sum_{k \in \Lambda} A_k (f_{L_k})_{\alpha_k+1} \) related to \( \{\varepsilon^{\alpha_k}L_k(\varepsilon); k \in \Lambda\} \). One can easily prove that in this case

\[
f' q.e. \sum_{k \in \Lambda} A_k (f'_{L_k})_{\alpha_k+1} [\varepsilon^{\alpha_k}L_k(\varepsilon)].
\]

**Proposition 2.15.** Let \( f \in F'(\mathbb{R}) \). If

\[
f q.e. \sum_{k \in \Lambda} A_k (f_{L_k})_{\alpha_k+1} [\varepsilon^{\alpha_k}L_k(\varepsilon)]
\]
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and

$$f \overset{q.e.}{\sim} \sum_{k \in \Lambda} \tilde{A}_k(f_L)_{\tilde{\alpha}_k+1} \{ \varepsilon^{\tilde{\alpha}_k} \tilde{L}_k(\varepsilon) \},$$

then $\alpha_k = \tilde{\alpha}_k$, $A_k = \tilde{A}_k$ and $L_k(\varepsilon) \sim \tilde{L}_k(\varepsilon)$, $\varepsilon \to 0^+$, $k \in \Lambda$.

**Proof.** Since $1 \in \Lambda$, by the properties of the quasi-asymptotics at $0^+$, $\alpha_1 = \tilde{\alpha}_1$, $A_1 = \tilde{A}_1$ and $L_1(\varepsilon) \sim \tilde{L}_1(\varepsilon)$, $\varepsilon \to 0^+$. If

$$A_1(f_{L_1})_{\alpha_1+1} + A_2(f_{L_2})_{\alpha_2+1} + R_2 = A_1(f_{\tilde{L}_1})_{\tilde{\alpha}_1+1} + \tilde{A}_2(f_{\tilde{L}_2})_{\tilde{\alpha}_2+1} + \tilde{R}_2,$$

where

$$\lim_{\varepsilon \to 0^+} \frac{R_2(\varepsilon x)}{\varepsilon^{\alpha_2} L_2(\varepsilon)} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \frac{\tilde{R}_2(\varepsilon x)}{\varepsilon^{\tilde{\alpha}_2} \tilde{L}_2(\varepsilon)} = 0 \quad \text{in} \quad F'(\mathbb{R}).$$

Since $\alpha_2 > \alpha_1$ and $\tilde{\alpha}_2 > \tilde{\alpha}_1$, the assumption $\alpha_2 \neq \tilde{\alpha}_2$ gives a contradiction. In the same way $L_2(\varepsilon) \sim \tilde{L}_2(\varepsilon)$, $\varepsilon \to 0^+$ and thus $A_2 = \tilde{A}_2$. The rest follows by induction. \hfill \Box

We give a structural proposition in $S'_+$.

**Proposition 2.16.** If $f \in S'_+$, then for each $m \in \Lambda$ there is a $p_m \in \mathbb{N}_0$ and a continuous function $F_m$ with $\text{supp } F_m \subset [0, \infty)$ such that

$$f = \left( F_m + \sum_{k=1}^{m} A_k f_{\alpha_k+1+1} \right)^{(p_m)} \quad \text{and} \quad \lim_{x \to 0^+} \frac{F_m(x)}{x^{\alpha_m+p_m} L_m(x)} = 0.$$

**Proof.** Observe that a version of Theorem 2.2 holds at $0^+$. Now, let $s \leq m$. By Definition 2.10

$$\left( f(\varepsilon x) - \sum_{k=1}^{m} A_k f_{\alpha_k+1}(\varepsilon x) \right) \frac{1}{\varepsilon^{\alpha_m} L_m(\varepsilon)}, \varphi(x) \to 0, \quad \text{as} \quad \varepsilon \to 0^+, \quad \varphi \in \mathcal{S}(\mathbb{R}),$$

and the claim follows. \hfill \Box

Other definitions and results of the quasi-asymptotic expansions of distributions are given in [52], [53], [55], [102], [118], [194], and [205].

We end this section with an important example of quasi-asymptotic expansion at $\infty$.
Example 3. The quasi-asymptotic expansion at $\infty$

$$f(kx) \sim \sum_{j=0}^{\infty} \frac{(-1)^j \mu_j}{j! k^{j+1}} (j)(x),$$

is called the Estrada-Kanwal moment asymptotic expansion.

Estrada and Kanwal have extensively studied this expansion in several distribution spaces as well as its numerous applications. For example, it holds if $f$ has compact support, and actually not just in the space $D'$ but in $E'$. The constants $\mu_j$ are the moments of $f$, namely, they are determined by $\mu_j = \langle f(x), x^j \rangle$. Observe that the moment asymptotic expansion gives a clear explanation of why the quasi-asymptotics at infinity is not a local property. For theory and applications of this interesting and important asymptotic expansion, we refer to [47], [52], [53], and [54].

2.10 The structure of quasi-asymptotics. Up-to-date results in one dimension

In this section we present results related to quasi-asymptotics in $D'(\mathbb{R})$ and $S'(\mathbb{R})$. We give complete answers to some questions started in the previous sections (cf. 2.6, 2.7 and 2.8). We shall follow the exposition from [171], [172], [173] and [186].

Our first aim is to describe the structure of (one-dimensional) quasi-asymptotics by means of complete structural theorems; this will be done in 2.10.3 and 2.10.5. The key tool for obtaining such results is the concepts of asymptotically and associate asymptotically homogeneous functions; we discuss these classes of functions in detail in 2.10.2 and 2.10.4.

We should employ a new notation for quasi-asymptotics at 0 and $\pm\infty$, which is more convenient for the purposes of this section. In order to emphasize the role of the slowly varying function, we will use the following notation for quasi-asymptotics,

$$f(\lambda x) = L(\lambda) g(\lambda x) + o(\lambda^\alpha L(\lambda)) \quad \text{in } F'(\mathbb{R}),$$

where the parameter $\lambda$ is taken to either zero or infinity. We call $\alpha$ the degree of the quasi-asymptotic behavior. Observe that $g$ may be identically zero, and all the results presented in this section are applicable to this situation as well. Recall that if (2.41) is satisfied and $F' = D'$, then $g$ is automatically homogeneous of degree $\alpha$ and it therefore has the form (2.3), depending on whether $\alpha \in \mathbb{R} \setminus (-\mathbb{N})$ or not.
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Let $g_j \in \mathcal{F}'(\mathbb{R})$ and let $\rho_j$ be arbitrary measurable functions, $j = 1, 2, \ldots, n$, we write

$$f(\lambda x) = \sum_{j=1}^{n} \rho_j(\lambda) g_j(x) + o(\rho_n(\lambda)) \text{ in } \mathcal{F}'(\mathbb{R}),$$

if

$$\langle f(\lambda x), \phi(x) \rangle = \sum_{j=1}^{n} \rho_j(\lambda) \langle g_j(x), \phi(x) \rangle + o(\rho_n(\lambda)), \quad \phi \in \mathcal{F}(\mathbb{R}).$$

2.10.1 Remarks on slowly varying functions

In this section we collect some results about slowly varying functions to be used in the future.

Let us assume that $L$ is a slowly varying function at the origin (cf. 0.3). Similar considerations are applicable for slowly varying functions at infinity.

Our first obvious observation is that only the behavior of $L$ near 0 plays a role in (2.41), and so we may impose to $L$ any behavior we want in intervals of the form $[A, \infty)$. Moreover, if $\tilde{L}$ is any slowly varying function which satisfies

$$\lim_{x \to 0^+} \frac{\tilde{L}(x)}{L(x)} = 1,$$

we may replace $L$ by $\tilde{L}$ in any statement about quasi-asymptotics without losing generality.

One of the most basic (and most important) results in the theory of slowly varying functions is the representation formula (see first two pages of [148]). Furthermore, the representation formula completely characterizes all the slowly varying functions; $L$ is slowly varying at the origin if and only if there exist measurable functions $u$ and $w$ defined on some interval $(0, B]$, $u$ being bounded and having a finite limit at 0 and $w$ being continuous on $[0, B]$ with $w(0) = 0$, such that

$$L(x) = \exp \left( u(x) + \int_{x}^{B} \frac{w(t)}{t} \, dt \right), \quad x \in (0, B].$$

This formula is important because it enables us to obtain some estimates on $L$. Since we are looking for suitable modifications of $L$, our first remark is that we can always assume that $L$ is defined in the whole $(0, \infty)$ and $L$ is
everywhere positive. This is shown by extending \( u \) and \( w \) to \((0, \infty)\) in any way we want.

Given any fixed \( \sigma > 0 \), then, by modifying \( u \) and \( w \), we can assume, when it is convenient, that \( B = 1 \), \( u \) is bounded on all over \((0, \infty)\) and \( |w(x)| < \sigma, x \in (0, \infty) \). In particular, we obtain the estimate

\[
\tilde{M} \min \left\{ x^{-\sigma}, x^\sigma \right\} < \frac{L(\lambda x)}{L(\lambda)} < M \max \left\{ x^{-\sigma}, x^\sigma \right\}, \quad \forall x, \lambda \in (0, \infty),
\]

for some positive constants \( M \) and \( \tilde{M} \). This result is known as Potter’s estimate [9], p. 25, and will be of vital importance in our investigations of the structural properties for quasi-asymptotics. Under the assumption of the last estimate we can use Lebesgue’s dominated convergence theorem in

\[
\int_0^\infty \left( \frac{L(\lambda x)}{L(\lambda)} - 1 \right) \phi(x) dx,
\]

for \( \phi \in \mathcal{S}(\mathbb{R}) \), to deduce that

\[
L(\lambda x)H(x) = L(\lambda)H(x) + o(L(\lambda)), \quad \text{in } \mathcal{S}'(\mathbb{R}).
\]

The reader should keep in mind (2.42) and (2.43), since from now on they will be implicitly used without any further reference, especially for differentiating asymptotic expressions in the future sections. We finally comment a well-known fact [9], [148]: As soon as \( L(ax) \sim L(x) \) holds for each \( a > 0 \), it automatically holds uniformly for \( a \) in compact subsets of \((0, \infty)\).

### 2.10.2 Asymptotically homogeneous functions

We study some properties of asymptotically homogeneous functions which will be applied later to the structural study of quasi-asymptotics. Let us proceed to define this class of functions.

**Definition 2.11.** A function \( b \) is said to be asymptotically homogeneous of degree \( \alpha \) at the origin (resp. at infinity) with respect to the slowly varying function \( L \), if it is measurable and defined in some interval \((0, A)\) (resp. on \((A, \infty)\)), \( A > 0 \), and for each \( a > 0 \),

\[
b(ax) = a^\alpha b(x) + o(L(x)), \quad x \to 0^+ \quad (\text{resp. } x \to \infty).
\]

Obviously, asymptotically homogeneous functions at the origin and at infinity are connected by the change of variables \( x \leftrightarrow x^{-1} \); therefore, most
of the properties of the class of asymptotically homogeneous functions at infinity can be obtained from those of the corresponding class at the origin.

Observe that no uniformity with respect to $a$ is assumed in Definition 2.11; however, the definition itself forces (2.44) to hold uniformly for $a$ on compact subsets. Indeed, we will show this fact by using a classical argument of J. Korevaar, T. van Aardenne Ehrenfest and N. G. de Bruijn [83], [9], [148] and [186].

Lemma 2.2. Let $b$ be asymptotically homogeneous of degree $\alpha$ with respect to $L$. Then, the relation

$$b(ax) = a^\alpha b(x) + o(L(x)),$$

holds uniformly for $a$ in compact subsets of $(0, \infty)$.

Proof. We show the assertion at the origin, the case at infinity can be obtained by the change of variables $x \leftrightarrow x^{-1}$. So assume that $b$ is asymptotically homogeneous function of degree $\alpha$ at the origin with respect to $L$. We may assume that $b$ is defined on $(0, 1]$. We rather work with the functions $c(x) = e^{\alpha x}b(e^{-x})$ and $s(x) = L(e^{-x})$, hence $c$ and $s$ are defined in $[0, \infty)$. By using a linear transformation between an arbitrary compact subinterval of $[0, \infty)$ and $[0, 1]$, it is enough to show that

$$c(h + x) - c(x) = o(e^{\alpha x} s(x)), \quad x \to \infty,$$

uniformly for $h \in [0, 1]$. Suppose that (2.45) is false. Then, there exist $0 < \varepsilon < 1$, a sequence $\langle h_m \rangle_{m=1}^\infty \in [0, 1]^\mathbb{N}$ and an increasing sequence of real numbers $\langle x_m \rangle_{m=1}^\infty$, $x_m \to \infty$, $m \to \infty$, such that

$$|c(h_m + x_m) - c(x_m)| \geq \varepsilon e^{\alpha x_m} s(x_m), \quad m \in \mathbb{N}. \quad (2.46)$$

Define, for $n \in \mathbb{N}$,

$$A_n = \left\{ h \in [0, 2]; |c(h + x_m) - c(x_m)| < \frac{\varepsilon}{3} e^{\alpha x_m} s(x_m), m \geq n \right\},$$

$$B_n = \left\{ h \in [0, 2]; |c(h + x_m + h_m) - c(h_m + x_m)| < \frac{\varepsilon}{3} e^{\alpha x_m} s(x_m + h_m), m \geq n \right\}.$$

Note that

$$[0, 2] = \bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n,$$

so we can select $N$ such that $\mu(A_n), \mu(B_n) > \frac{1}{2}$ (here $\mu(\cdot)$ stands for Lebesgue measure), for all $n \geq N$. For each $n \in \mathbb{N}$, put $C_n = \{h_n\} + B_n$. 

Then, we have \( \mu(C_n) > \frac{3}{2}, n \geq N \), and \( C_n, A_n \subseteq [0, 3] \). It follows that \( A_n \cap C_n \neq \emptyset, n > N \). For each \( n \geq N \), select \( u_n \in A_n \cap C_n \). In particular, we have \( u_n - h_n \in B_n \), and hence,

\[
|c(u_n + x_n) - c(x_n)| < \frac{\varepsilon}{3} e^{\alpha x_n} s(x_n),
\]

\[
|c(u_n + x_n) - c(x_n + h_n)| < \frac{\varepsilon}{3} e^{\alpha x_n} s(x_n + h_n)
\]

which implies that for all \( n \geq N \),

\[
|c(x_n + h_n) - c(x_n)| < \frac{\varepsilon}{3} e^{\alpha x_n} (s(x_n) + s(x_n + h_n)).
\]

Using that \( s(x + h) - s(x) = o(s(x)), x \to \infty \), uniformly for \( h \) on compact subsets of \((0, \infty)\), we have that for all \( n \) sufficiently large, \( s(x_n + h_n) \leq 2s(x_n) \), which implies that for \( n \) big enough

\[
|c(x_n + h_n) - c(x_n)| < \varepsilon e^{\alpha x_n} s(x_n),
\]

in contradiction to (2.46). Therefore, (2.45) must hold uniformly for \( h \in [0, 1] \).

\[\square\]

**Corollary 2.1.** If \( b \) is asymptotically homogeneous at the origin (resp. at infinity) with respect to a slowly varying function, then \( b \) is locally bounded in some interval of the form \((0, B)\) (resp. \((B, \infty)\)).

**Proof.** It follows directly from Lemma 2.2. \[\square\]

We now obtain the behavior of asymptotically homogeneous functions when the degree is not a negative integer.

**Theorem 2.23.** Suppose that \( b \) is asymptotically homogeneous at the origin (resp. at infinity) with respect to the slowly varying function \( L \). Assume that its degree is not a negative integer. Then

(i) If \( \alpha > 0 \) (resp. \( \alpha < 0 \) for the case at infinity), then

\[
b(x) = o(L(x)), x \to 0^+ \text{ (resp. } x \to \infty) .
\]

(ii) If \( \alpha < 0 \) (resp. \( \alpha > 0 \)), then there exists a constant \( \gamma \) such that

\[
b(x) = \gamma x^\alpha + o(L(x)), x \to 0^+ \text{ (resp. } x \to \infty) .
\]
Proof. We show only the assertion at the origin, the case at infinity follows again from a change of variables.

Let us first show i). Assume that \( \alpha > 0 \). Let \( 0 < \eta \) be any arbitrary number. We keep \( \eta < 2^{\alpha} - 1 \). Let \( x_0 > 0 \) such that

\[
\left| b \left( \frac{x}{2} \right) - 2^{-\alpha} b(x) \right| \leq \eta L(x) \quad \text{and} \quad |L(2x) - L(x)| \leq \eta L(x), \ 0 < x < x_0,
\]

and \( M = \sup \{ |b(x)|/L(x); \frac{1}{2} \leq x \leq x_0 \} < \infty \). Let \( x \in [x_0/2, x_0] \). We obtain from (2.49),

\[
\left| \frac{b(x/2^n)}{L(x/2^n)} \right| \leq 2^{-\alpha n} \frac{|b(x)|}{L(x/2^n)} + \eta \sum_{j=0}^{n-1} 2^{-\alpha(n-1-j)} \frac{L(x/2^j)}{L(x/2^n)}.
\]

Thus, with \( t = x/2^n \), and \( t \in [x_0/2^{n+1}, x_0/2^n] \),

\[
\left| \frac{b(t)}{L(t)} \right| \leq 2^{-\alpha n} \frac{|b(x)|}{L(x/2^n)} + \eta \sum_{j=0}^{n-1} 2^{-\alpha(n-1-j)} \frac{L(2^{j+1}t)}{L(t)}.
\]

By this and

\[
L(2^{j+1}t)/L(2^j t) \leq (1 + \eta), \ j = 0, \ldots, n - 1,
\]

we have that if \( t \in \left[ 2^{-(n+1)}x_0, 2^{-n}x_0 \right] \), then

\[
\left| \frac{b(t)}{L(t)} \right| \leq M \left( \frac{1 + \eta}{2^\alpha} \right)^n + \eta \left( 1 + \eta \right) \sum_{j=0}^{\infty} \left( \frac{1 + \eta}{2^\alpha} \right)^j = M \left( \frac{1 + \eta}{2^\alpha} \right)^n + \eta \left( 1 + \eta \right) \frac{2^\alpha}{2^\alpha - 1 - \eta}.
\]

Let us prove that for every \( \varepsilon > 0 \) there exists a positive \( \sigma \) such that \( |b(t)/L(t)| < \varepsilon, \ t \in (0, \sigma) \). First, we have to take \( \eta \), small enough, such that

\[
\eta \left( 1 + \eta \right) \frac{2^\alpha}{2^\alpha - 1 - \eta} < \frac{\varepsilon}{2}
\]

and \( n_0 \in \mathbb{N} \) such that

\[
M \left( \frac{1 + \eta}{2^\alpha} \right)^n < \frac{\varepsilon}{2}, \ n \geq n_0.
\]

Then, it follows that \( |b(t)/L(t)| < \varepsilon, \ t \in (0, \sigma) \), if we take \( \sigma = x_0/2^{n_0} \). This completes the first part of the proof.
We now show (ii). Assume that \( \alpha < 0 \). We rather work with 
\[ c(x) = e^{\alpha x} b(e^{-x}) \] 
and 
\[ s(x) = L(e^{-x}) \]
Then \( c \) satisfies
\[ c(h + x) - c(x) = o(e^{\alpha x} s(x)) \], \( x \to \infty \),
uniformly for \( h \in [0, 1] \). Given \( \varepsilon > 0 \), we can find \( x_0 > 0 \) such that for all \( x > x_0 \) and \( h \in [0, 1] \),
\[ |c(x + h) - c(x)| \leq \varepsilon e^{\alpha x} s(x) \quad \text{and} \quad |s(h + x) - s(x)| \leq (e^{-\frac{x}{2}} - 1) s(x). \]
So we have that
\[ |c(h + n + x) - c(x)| \leq |c(h + n + x) - c(n + x)| + |c(n + x) - c(x)| \leq \varepsilon e^{\alpha (n + x)} s(n + x) + \sum_{j=0}^{n-1} |c(j + 1 + x) - c(j + x)| \leq \varepsilon e^{\alpha x} s(x) \sum_{j=0}^{n-1} e^{\alpha j} \leq \varepsilon e^{\alpha x} s(x) \frac{1}{1 - e^{\alpha}}, \]
where the last estimate follows from \( s(x + j) \leq s(x) e^{-\alpha j}/2 \). Since \( s(x) = o(e^{-\alpha x}) \) as \( x \to \infty \), it shows that there exists \( \gamma \in \mathbb{R} \) such that
\[ \lim_{x \to \infty} c(x) = \gamma. \]
Moreover, the estimate shows that
\[ c(x) = \gamma + o(e^{\alpha x} s(x)) \], \( x \to \infty \),
thus, changing the variables back, we have obtained,
\[ b(x) = \gamma x^\alpha + o(L(x)) \], \( x \to 0^+ \).

We remark that (2.47) and (2.48) trivially imply that \( b \) is asymptotically homogeneous of degree \( \alpha \) with respect to \( L \).

Asymptotically homogeneous functions of degree zero have a more complex asymptotic behavior. For example if \( L \equiv 1 \), any asymptotically homogeneous function is the logarithm of a slowly varying function. Instead of attempting to find their behavior in the classical sense, we will study their distributional behavior.
Lemma 2.3. Let \( b \) be asymptotically homogeneous of degree 0 at the origin (respectively at infinity) with respect to the slowly varying function \( L \). If \( \sigma < 0 \) (resp. \( \sigma > 0 \)) then,
\[
b(x) = o(x^{\sigma}), \quad x \to 0^+ \quad \text{(resp. } x \to \infty)\text{.}
\]
In particular, \( b(x)(L(x))^{-1} \) is integrable near the origin (resp. locally integrable near \( \infty \)).

Proof. We know that \( L(x) = o(x^{\sigma}) \). Then for each \( a > 0 \), \( b(ax) = b(x) + o(x^{\sigma}) \) and this implies that \( x^{-\sigma}b(x) \) is asymptotically homogeneous of degree \( -\sigma \) with respect to the constant function 1. From (i) of Theorem 2.23, it follows that \( b(x) = o(x^{\sigma}) \). \( \square \)

We now describe the behavior of asymptotically homogeneous functions of degree zero at the origin. The next theorem will be very important in the next section.

Theorem 2.24. Let \( b \) be asymptotically homogeneous of degree zero at the origin with respect to the slowly varying function \( L \). Suppose that \( b \) is integrable on \( (0,A) \). Then
\[
b(\varepsilon x)(H(x) - H(\varepsilon x - A)) = b(\varepsilon)H(x) + o(L(\varepsilon)) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}),
\]
where \( H \) is the Heaviside function.

Proof. Let \( \phi \in \mathcal{D}(\mathbb{R}) \). Find \( B \) such that \( \text{supp } \phi \subseteq [-B,B] \), then there exists \( \varepsilon_\phi < 1 \) such that
\[
(b(\varepsilon x), \phi(x)) = \int_0^B b(\varepsilon x)\phi(x)dx = \int_0^B b(\varepsilon x)\phi(x)dx, \quad \varepsilon < \varepsilon_\phi. \tag{2.51}
\]
Replacing \( \phi(x) \) by \( B\phi(Bx) \) and \( \varepsilon_\phi \) by \( B\varepsilon_\phi \), we may assume that \( B = 1 \). Our aim is to show that for some \( \varepsilon_0 < 1 \),
\[
\frac{b(\varepsilon x) - b(\varepsilon)}{L(\varepsilon)}, \quad x \in (0,1], \quad \varepsilon < \varepsilon_0,
\]
is dominated by an integrable function in \( (0,1] \) for the use of the Lebesgue theorem. For this goal, we assume that \( L \) satisfies the following estimate,
\[
\frac{L(\varepsilon x)}{L(\varepsilon)} \leq Mx^{-\frac{1}{2}}, \quad x \in (0,1], \quad \varepsilon \in (0,\varepsilon_\phi). \tag{2.52}
\]
By Lemma 2.2, there exists \( 0 < \varepsilon_0 < \varepsilon_\phi \) such that
\[
|b(\varepsilon x) - b(\varepsilon)| < L(\varepsilon), \quad x \in [1/2,2], \quad \varepsilon < \varepsilon_0.
\]
We keep $\varepsilon < \varepsilon_0$ and $x \in [2^{-n-1}, 2^{-n}]$. Then

$$|b(\varepsilon x) - b(\varepsilon)| \leq |b(2\varepsilon x) - b((2\varepsilon)/2)| + |b(2\varepsilon x) - b(\varepsilon)|$$

$$\leq L(2\varepsilon x) + |b(2\varepsilon x) - b(\varepsilon)| \leq \sum_{i=1}^{n} L(2^i\varepsilon x) + L(\varepsilon)$$

$$\leq \sum_{i=1}^{n} (2^i x)^{-1/2} L(\varepsilon) + L(\varepsilon),$$

where the last inequality follows from (2.52). Then, if $\varepsilon < \varepsilon_0$ and $x \leq 1$, then

$$\frac{|b(\varepsilon x) - b(\varepsilon)|}{L(\varepsilon)} \leq M_1 x^{-\frac{1}{2}} + 1,$$

where $M_1 = M(\sqrt{2} + 1)$. Therefore we can apply Lebesgue’s dominated convergence theorem to deduce (2.50). \square

We also have a similar result at infinity, this fact is stated in the next theorem. Since its a corollary of Theorem 2.27, we omit its proof and refer the reader to 2.10.4.

**Theorem 2.25.** Let $b$ be asymptotically homogeneous of degree zero at infinity with respect to the slowly varying function $L$. Suppose that $b$ is locally integrable on $[A, \infty)$. Then

$$b(\lambda x)H(\lambda x - A) = b(\lambda)H(x) + o(L(\lambda)) \quad \text{as} \quad \lambda \to \infty \quad \text{in} \quad S'(\mathbb{R}).$$

2.10.3 *Relation between asymptotically homogeneous functions and quasi-asymptotics*

We introduced asymptotically homogeneous functions in order to study the structure of the quasi-asymptotics for Schwartz distributions. We now derive structural theorems for quasi-asymptotics in some cases from the fundamental properties of this class of functions (Theorems 2.23, 2.24 and 2.25).

The technique to be employed here is based on the analysis of the parametric coefficients resulting after performing several integrations of the quasi-asymptotic behavior, these coefficients are naturally connected with the class of asymptotically homogeneous functions. The technique of integration of distributional asymptotic relations goes back to the classical
work of Lojasiewicz [93, 116] (cf. Theorem 2.19 in 2.8). Later on, the
properties of the parametric coefficients were singled out and recognized
as asymptotically homogeneous functions in [45], [171], [172], [173], [176],
[186].

The next proposition provides the intrinsic link between quasi-asympto-
tics and asymptotically homogeneous functions.

**Proposition 2.17.** Let $f \in \mathcal{D}'(\mathbb{R})$ have quasi-asymptotic behavior in
$\mathcal{D}'(\mathbb{R})$

$$ f(\lambda x) = L(\lambda)g(\lambda x) + o(\lambda^\alpha L(\lambda)) \quad \text{as } \lambda \to \infty \text{ (resp. } \lambda \to 0^+) , \quad (2.53) $$

where $L$ is a slowly varying function and $g$ is a homogeneous distribution
degree $\alpha \in \mathbb{R}$. Let $n \in \mathbb{N}$. Suppose that $g$ admits a primitive of order $n,$
that is, $G_n \in \mathcal{D}'(\mathbb{R})$ and $G_n^{(n)} = g$, which is homogeneous of degree $n + \alpha$.
Then, for any given $F_n$, an $n$-primitive of $f$ in $\mathcal{D}'(\mathbb{R})$, there exist functions
$b_0, \ldots, b_{n-1}$, continuous on $(0, \infty)$, such that

$$ F_n(\lambda x) = L(\lambda)G_n(\lambda x) + \sum_{j=0}^{n-1} \lambda^{n-j} b_j(\lambda) \frac{x^{n-1-j}}{(n-j)!} + o(\lambda^{n+1} L(\lambda)) \quad (2.54) $$
as $\lambda \to \infty$ (resp. $\lambda \to 0^+$) in $\mathcal{D}'(\mathbb{R})$, where each $b_j$ is asymptotically
homogeneous of degree $-\alpha - j - 1$.

**Proof.** Recall that any $\phi \in \mathcal{D}(\mathbb{R})$ is of the form

$$ \phi = C_\phi \phi_0 + \theta', \text{ where } C_\phi = \int_{-\infty}^{\infty} \phi(t)dt, \theta \in \mathcal{D}(\mathbb{R}), \quad (2.55) $$

and $\phi_0 \in \mathcal{D}(\mathbb{R})$ is chosen so that $\int_{-\infty}^{\infty} \phi_0(t)dt = 1$. The evaluations of
primitives $F_1$ of $f$ and $G_1$ of $g$ on $\phi$ are given by

$$ \langle F_1, \phi \rangle = C_\phi \langle F_1, \phi_0 \rangle - \langle f, \theta \rangle \quad \text{and} \quad \langle G_1, \phi \rangle = C_\phi \langle G_1, \phi_0 \rangle - \langle g, \theta \rangle. $$

This implies

$$\left\langle \frac{F_1(\lambda x)}{\lambda^{\alpha+1} L(\lambda)}, \phi(x) \right\rangle = C_\phi \left\langle \frac{F_1(\lambda x)}{\lambda^{\alpha+1} L(\lambda)}, \phi_0(x) \right\rangle - \left\langle \frac{f(\lambda x)}{\lambda^{\alpha+1} L(\lambda)}, \theta(x) \right\rangle, \quad (2.56) $$

and

$$ \left\langle \frac{G_1(\lambda x)}{\lambda^{\alpha+1} L(\lambda)}, \phi(x) \right\rangle = C_\phi \left\langle \frac{G_1(\lambda x)}{\lambda^{\alpha+1} L(\lambda)}, \phi_0(x) \right\rangle - \left\langle \frac{g(\lambda x)}{\lambda^{\alpha+1} L(\lambda)}, \theta(x) \right\rangle. \quad (2.57) $$

With $c_0(\lambda) = \langle (F_1 - G_1)(\lambda x), \phi_0(x) \rangle$, $\lambda \in (0, \infty)$, from (2.53), it follows

$$ F_1(\lambda x) = L(\lambda)G_1(\lambda x) + c_0(\lambda) + o(\lambda^{\alpha+1} L(\lambda)) \quad \text{in } \mathcal{D}'(\mathbb{R}). \quad (2.58) $$
So relation (2.54) follows by induction from (2.58) and (2.53).

We shall now concentrate in showing the property of the $b_j$’s. We set $F_m = F_m^{(n-m)}$ and $G_m = G_m^{(n-m)}$, $m \in \{1, \ldots, n\}$. By differentiating relation (2.54) $(n-m)$-times, it follows that

$$F_m(\lambda x) = L(\lambda)G_m(\lambda x) + \sum_{j=0}^{m-1} \lambda^{\alpha+m}b_j(\lambda) \frac{x^{m-1-j}}{(m-1-j)!} + o(\lambda^{\alpha+m}L(\lambda))$$

(2.59)
in $\mathcal{D}'(\mathbb{R})$. Choose $\phi \in \mathcal{D}(\mathbb{R})$ such that $\int_{-\infty}^{\infty} \phi(x)x^j dx = 0$ for $j = 1, \ldots, m-1$, and $\int_{-\infty}^{\infty} \phi(x)dx = 1$. Then evaluating (2.59) at $\phi$, we have that

$$(a\lambda)^{\alpha+m}b_{m-1}(a\lambda) + L(a\lambda) \langle G_m(a\lambda x), \phi(x) \rangle + o(\lambda^{\alpha+m}L(\lambda))$$

$$= \langle F_m(a\lambda x), \phi(x) \rangle = \frac{1}{a} \langle F_m(\lambda x), \phi \left( \frac{x}{a} \right) \rangle$$

$$= \lambda^{\alpha+m}b_{m-1}(\lambda) + L(\lambda) \langle G_m(a\lambda x), \phi(x) \rangle + o(\lambda^{\alpha+m}L(\lambda)) ,$$

and so, with $j = m-1 \in \{0, \ldots, n-1\}$, for each $a > 0$,

$$b_j(a\lambda) = a^{-\alpha-j-1}b_j(\lambda) + o(L(\lambda)).$$

With the aid of asymptotically homogeneous functions, we can now obtain our first structural theorem. Observe that for the case at $\pm \infty$ we recover Theorem 2.14, while at 0 we actually extend Theorem 2.19 and Theorem 2.20 (cf. Remark 2 in 2.8).

**Theorem 2.26.** Let $f \in \mathcal{D}'(\mathbb{R})$ have quasi-asymptotic behavior at $\pm \infty$ (resp. at the origin) in $\mathcal{D}'(\mathbb{R})$,

$$f(\lambda x) = C_- L(\lambda) \frac{(\lambda x)^\alpha}{\Gamma(\alpha + 1)} + C_+ L(\lambda) \frac{(\lambda x)^\alpha}{\Gamma(\alpha + 1)} + o(\lambda^\alpha L(\lambda)) .$$

(2.60)

If $\alpha \notin \{-1, -2, \ldots\}$, then there exist a non-negative integer $m > -\alpha - 1$ and an $m$-primitive $F$ of $f$ such that $F$ is continuous (resp. continuous near 0) and

$$\lim_{x \to \pm \infty} \frac{\Gamma(\alpha + m + 1)F(x)}{x^m |x|^\alpha L(|x|)} = C_\pm \quad (\text{resp.} \quad \lim_{x \to 0^\pm}).$$

(2.61)

Conversely, if these conditions hold, then (by differentiation) (2.60) follows.
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Proof. On combining Proposition 2.17 and Theorem 2.23, one obtains that for each $n \in \mathbb{N}$ and $F_n$, an $n$-primitive of $f$, there exist constants $\gamma_0, \ldots, \gamma_{n-1}$ such that in the sense of convergence in $\mathcal{D}'(\mathbb{R})$,

$$F_n(\lambda x) = \sum_{j=0}^{n-1} \frac{(\lambda x)^j}{j!} + C_+ \frac{(-1)^n L(\lambda)(\lambda x)^{\alpha+n}}{\Gamma(\alpha+n+1)} + C_- \frac{L(\lambda)(\lambda x)^{\alpha+n}}{\Gamma(\alpha+n+1)} + o\left((\lambda^\alpha L(\lambda))^{n+1}\right).$$

(2.62)

It follows from the convergence $\mathcal{D}'(\mathbb{R})$ that there is $m \in \mathbb{N}$, sufficiently large, such that any $m$-primitive of $f$ is continuous and (2.62) holds (with $n = m$) uniformly for $x \in [-1, 1]$. Pick a specific $m$-primitive of $f$, say $F_m$, then from (2.62) there is a polynomial $p$ of degree at most $m-1$ such that

$$F_m(\lambda x) = p(\lambda x) + C_- L(\lambda) \frac{(-1)^m (\lambda x)^{\alpha+m}}{\Gamma(\alpha+m+1)} + C_+ L(\lambda) \frac{(\lambda x)^{\alpha+m}}{\Gamma(\alpha+m+1)} + o\left((\lambda^\alpha L(\lambda))^{m+1}\right),$$

uniformly for $x \in [-1, 1]$. Then setting $F = F_m - p$, $x = 1, -1$ and replacing $\lambda$ by $x$, relation (2.61) follows at once. The converse follows by differentiation and the properties of regularly varying functions.

We now start to analyze quasi-asymptotics of negative integral degrees. In this section we will only study the quasi-asymptotics $f(\lambda x) \sim \gamma L(\lambda)\delta(\lambda x)$. We postpone the general case for 2.10.5, after the introduction of associate asymptotically homogeneous function in 2.10.4.

**Proposition 2.18.** Let $f \in \mathcal{D}'(\mathbb{R})$ have quasi-asymptotic behavior at $\pm \infty$ (at the origin) in $\mathcal{D}'(\mathbb{R})$,

$$f(\lambda x) = \gamma L(\lambda)\delta(\lambda x) + o\left((\lambda^{-1} L(\lambda))^{m-1}\right) \quad \text{as} \quad \lambda \to \infty \quad (\text{resp.} \quad \lambda \to 0^+).$$

(2.63)

Then, there exist $m \in \mathbb{N}$, a function $b$, being asymptotically homogeneous of degree 0 with respect to $L$, and an $(m+1)$-primitive $F$ of $f$ such that $F$ is continuous (resp. continuous near 0) and

$$F(x) = \gamma L(|x|) \frac{x^m}{2m!} \text{sgn} x + b(|x|) \frac{x^m}{m!} + o(|x|^m L(|x|)).$$

(2.64)

Conversely, if (2.64) holds, then (2.63) follows by differentiation.

Proof. The existence of $m$, $b$, and $F$ satisfying (2.64) follows from the weak convergence of (2.63), Proposition 2.17 and Theorem 2.23, as in the proof of Theorem 2.26. The converse is shown by applying Theorem 2.25 (resp. Theorem 2.24) and differentiating $(m+1)$-times.  \(\square\)
2.10.4 Associate asymptotically homogeneous functions

We now introduce the main tool for the study of structural properties of quasi-asymptotics of negative integral degree. Associate asymptotically homogeneous functions are a generalization of asymptotically homogeneous functions. Let us define this class of functions.

**Definition 2.12.** A function \( b \) is said to be **associate asymptotically homogeneous of degree 0 at the origin** (resp. at infinity) with respect to the slowly varying function \( L \), if it is measurable and defined in some interval \((A, \infty)\), \( A > 0 \), and there exists a constant \( \beta \) such that for each \( a > 0 \),

\[
 b(ax) = b(x) + \beta L(x) \log a + o(L(x)) \quad \text{,} \quad x \to 0^+ \quad \text{(resp.} \quad x \to \infty). \tag{2.65}
\]

**Remark.** Associate asymptotically homogeneous functions are also known as de Haan functions (cf. [9] and [11]).

We may use the same argument employed in the proof of Lemma 2.2 to show uniform convergence.

**Lemma 2.4.** Relation (2.65) holds uniformly in compact subsets of \((0, \infty)\).

We shall study the distributional asymptotic properties of this class of functions in detail. We first roughly estimate the behavior of associate asymptotically homogeneous functions of degree 0.

**Lemma 2.5.** Let \( b \) be **associate asymptotically homogeneous of degree 0 at the origin** (resp. at infinity) with respect to \( L \), then for each \( \sigma < 0 \) (resp. \( \sigma > 0 \)),

\[
 b(x) = o(x^\sigma) \quad \text{,} \quad x \to 0^+ \quad \text{(resp.} \quad x \to \infty). \tag{2.66}
\]

**Proof.** We know that \( L(x) = o(x^\sigma) \), for each \( \sigma > 0 \) [148]. Hence \( b(ax) = b(x) + o(x^\sigma) \) and thus \( x^{-\sigma}b(x) \) is asymptotically homogeneous of degree \(-\sigma\) with respect to \( L \equiv 1 \), so (2.66) follows from Theorem 2.23. \( \square \)

The next two theorems will be crucial in 2.10.5. They generalize Theorems 2.24 and 2.25. We only give the proof at infinity, the proof at the origin is similar to that of Theorem 2.24.
Theorem 2.27. Let $b$ be a locally integrable associate asymptotically homogeneous function of degree zero at infinity with respect to the slowly varying function $L$. Suppose that $b$ is defined on $[A, \infty)$. Then
\[ b(\lambda x)H(\lambda x - A) = b(\lambda)L(\lambda)\lambda x + o(L(\lambda)) \quad \text{in } S'(\mathbb{R}), \] (2.67)
where $H$ is the Heaviside function.

Proof. Let $\lambda_0$ be any positive number. The function $b$ can be decomposed as $b = b_1 + b_2$, where $b_1 \in L^1(\mathbb{R})$ has compact support and $b_2(x) = b(x)H(x - \lambda_0)$ is associate asymptotically homogeneous function of degree zero at infinity. Since $b_1$ satisfies the moment asymptotic expansion (cf. Example 3 in 2.9), it follows that $b_1(\lambda x) = O(\lambda^{-1}) = o(L(\lambda))$ as $\lambda \to \infty$ in $S'(\mathbb{R})$. Therefore, we can always assume that $A = \lambda_0$, where $\lambda_0$ is selected at our convenience.

Our aim is to show that there is some $\lambda_0 > 1$ such that
\[ J(x, \lambda) := \phi(x) \frac{b(\lambda x) - b(\lambda) - \beta L(\lambda) \log x}{L(\lambda)} H(\lambda x - \lambda_0) \]
is dominated by an integrable function, whenever $\phi \in S(\mathbb{R})$, for the use of the Lebesgue dominated convergence theorem. For this goal, we can always assume that $L$ is positive everywhere and satisfies the following estimate (cf. 2.10.1)
\[ \frac{L(\lambda x)}{L(\lambda)} \leq M \max \left\{ x^{-\frac{4}{1}}, x^{\frac{2}{4}} \right\}, \quad x, \lambda \in (0, \infty), \] (2.68)
for some positive constant $M$. Because of the uniformity of (2.65) on compact sets, there exists a $\lambda_0 > 1$ such that
\[ |b(\lambda x) - b(\lambda) - \beta L(\lambda) \log x| < L(\lambda), \quad x \in [1, 2], \lambda_0 < \lambda. \]
Let $n$ be a positive integer. We keep $\lambda_0 < \lambda$ and $x \in [2^n, 2^{n+1}]$. Then
\[ |b(\lambda x) - b(\lambda) - \beta L(\lambda) \log x| \leq |b(\lambda x) - b(\lambda)| + |\beta| L(\lambda) \log x \]
\[ \leq |\beta| L(\lambda) \log x + |b(2\lambda x/2) - b(\lambda x/2) - \beta L(\lambda x/2) \log 2| \]
\[ + |\beta| L(\lambda x/2) \log 2 + |b(\lambda x/2) - b(\lambda)| \]
\[ \leq |\beta| L(\lambda) \log x + (1 + |\beta| \log 2) L(\lambda x/2) + |b(\lambda x/2) - b(\lambda)| \]
\[ \leq (1 + |\beta| \log 2) \sum_{j=1}^{n} L(2^{-j} \lambda x) + |\beta| L(\lambda) \log 2 x + L(\lambda) \]
\[ \leq \left( M x^{\frac{4}{1}} (1 + |\beta| \log 2) \sum_{j=1}^{n} (1/2)^{\frac{j}{4}} + |\beta| \log 2 x + 1 \right) L(\lambda), \]
where the last inequality follows from (2.68). So if \( \lambda_0 < \lambda \) and \( 1 < x \), then
\[
\frac{|b(\lambda x) - b(\lambda) - \beta L(\lambda) \log x|}{L(\lambda)} \leq M_1 x^{\frac{1}{4}},
\]
for some \( M_1 > 0 \). Now if \( \lambda_0/\lambda < x < 1 \), we have that
\[
\left| b(\lambda x) - b(\lambda) - \beta L(\lambda) \log x \right| \leq \left( 1 + \frac{L(\lambda x)}{L(\lambda)} \right) |\beta \log x| + \left( 1 + Mx^{-\frac{1}{4}} \right) |\beta \log x| + MMM_1 x^{-\frac{1}{2}}.
\]
Therefore \( J(x, \lambda) \) is dominated by an integrable function for \( \lambda > \lambda_0 \), so we apply Lebesgue dominated convergence theorem to deduce that
\[
\lim_{\lambda \to \infty} \int_0^\infty J(x, \lambda) dx = 0.
\]
Finally,
\[
\langle b(\lambda x)H(\lambda x - \lambda_0), \phi(x) \rangle - b(\lambda) \int_0^\infty \phi(x) dx - \beta L(\lambda) \int_0^\infty \log x \phi(x) dx = L(\lambda) \int_0^\infty J(x, \lambda) dx + L(\lambda)O \left( \frac{\log \lambda}{\lambda} \right) + O \left( \frac{b(\lambda)}{\lambda L(\lambda)} \right) = o(L(\lambda)) + L(\lambda)O \left( \frac{b(\lambda)}{\lambda L(\lambda)} \right) = o(L(\lambda)), \quad \lambda \to \infty,
\]
where in the last equality we have used Lemma 2.5 and the fact that slowly varying functions are \( o(\lambda^\sigma) \) for any \( \sigma > 0 \). This completes the proof of (2.67). \( \square \)

**Theorem 2.28.** Let \( b \) be a locally integrable associate asymptotically homogeneous function of degree zero at the origin with respect to the slowly varying function \( L \). Suppose that \( b \) is defined on \( (0, A] \). Then
\[
b(\varepsilon x)(H(x) - H(\varepsilon x - A)) = b(\varepsilon)H(x) + L(\varepsilon)\beta H(x) \log x + o(L(\varepsilon)), \quad (2.69)
\]
as \( \varepsilon \to 0^+ \) in \( \mathcal{D}'(\mathbb{R}) \).
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**Corollary 2.2.** Let $b$ be an associate asymptotically homogeneous function of degree 0 with respect to $L$. Then, there exists an associate asymptotically homogeneous function $c \in C^\infty[0, \infty)$ such that $b(x) = c(x) + o(L(x))$.

**Proof.** We may assume that $L \in C^\infty[0, \infty)$ [148], Section 1.4. Find $B$ such that $b$ is locally bounded in $[B, \infty)$, this can be done because of Lemma 2.4. Take $\phi \in D(\mathbb{R})$ such that $\int_0^\infty \phi(t) dt = 1$ and set $c(x) = \int_{B/x}^\infty b(xt) \phi(t) dt - \beta L(x) \int_0^\infty \phi(t) \log t dt$, the corollary now follows from Theorem 2.27 (resp. Theorem 2.28). □

We may also use Corollary 2.2 to obtain a representation formula for associate asymptotically homogeneous functions, this is the analog to [148], Theorem 1.2 for slowly varying functions.

**Theorem 2.29.** The function $b$ is associate asymptotically homogeneous of degree 0 at $\infty$ satisfying (2.65) if and only if there is a positive number $A$ such that

$$b(x) = \eta(x) + \int_A^x \frac{\tau(t)}{t} dt, \quad x \geq A,$$

(2.70)

where $\eta$ is a locally bounded measurable function on $[A, \infty)$ such that $\eta(x) = M + o(L(x))$ as $x \to \infty$, for some number $M$, and $\tau$ is a $C^\infty$-function such that $\tau(x) \sim \beta L(x)$ as $x \to \infty$.

**Proof.** Assume first that $b_1$ is $C^\infty$, defined on $[0, \infty)$ and satisfies that hypothesis of the theorem. We can find $L_1 \sim L$ which is $C^\infty$ and satisfies $xL_1(x) = o(L(x))$ as $x \to \infty$ [148], p. 7. Let $\phi$ and $c$ as in the proof of Corollary 2.2 corresponding to $b_1$ and $L_1$, additionally assume that supp $\phi \subseteq (0, \infty)$. From Theorem 2.27, we have that

$$b_1'(\lambda x) = \frac{b_1(\lambda)}{\lambda} \delta(x) + \beta \frac{L(x)}{x} \Psi(H(x)/x) + o\left(\frac{L(x)}{x}\right)$$

as $\lambda \to \infty$

in $S'(\mathbb{R})$, where $\Psi(H(x)/x) = (H(x) \log |x|)'$, since distributional asymptotics can be differentiated. Then, for $x$ positive

$$xc'(x) = x \int_0^\infty b_1'(xt) \phi(t) dt - \beta x L_1(x) \int_0^\infty \phi(t) \log t dt$$

$$= x \int_0^\infty b_1'(xt) \phi(t) dt + o(L(x))$$

$$= b_1(x) \cdot 0 + \beta L(x) \int_0^\infty \phi(t) dt + o(L(x))$$

$$= \beta L(x) + o(L(x)) \quad \text{as} \quad x \to \infty.$$
Set $\tau(x) = xc'(x)$. Find $A > 0$ such that $L$ is locally integrable on $[A, \infty)$, one has that $b_1(x) = c(A) + \int_A^x (\tau(t)/t) dt + o(L(x))$.

In the general case, let $A$ be a number such that $b$ and $L$ are locally bounded on $[A, \infty)$ and let $b_1$ the function from Corollary 2.2 such that $b(x) = b_1(x) + o(L(x))$, then we can apply the previous argument to $b_1$ to find $\tau$ as before, so we obtain (2.70) with $\eta(x) = b(x) - \int_A^x (\tau(t)/t) dt = c(A) + o(L(x))$. □

A change of variables $x \leftrightarrow x^{-1}$ in Theorem 2.29 implies the analog result at 0.

**Theorem 2.30.** The function $b$ is associate asymptotically homogeneous of degree 0 at the origin satisfying (2.65) if and only if there is a positive number $A$ such that

$$b(x) = \eta(x) + \int_x^A \frac{\tau(t)}{t} dt, \quad x \geq A,$$

where $\eta$ is a locally bounded measurable function on $[A, \infty)$ such that $\eta(x) = M + o(L(x))$ as $x \to 0^+$, for some number $M$, and $\tau$ is a $C^\infty$ function such that $\tau(x) \sim \beta L(x)$ as $x \to 0^+$.

A slightly different representation formula is given in [148], but, except for the smoothness of $\tau$, both are equivalent.

### 2.10.5 Structural theorems for negative integral degrees. The general case

This section is dedicated to the study of structural properties of quasi-asymptotics with negative integral degree, solving the question posed in Remark 3 of 2.8.

The next lemma reduces the analysis of negative integral degrees to the case of degree $-1$.

**Lemma 2.6.** Let $f \in \mathcal{D}'(\mathbb{R})$ and $k$ be a positive integer. Then $f$ has the quasi-asymptotic behavior in $\mathcal{D}'(\mathbb{R})$,

$$f(\lambda x) = \gamma \lambda^{-k} L(\lambda) \delta^{(k-1)}(x) + (-1)^{k-1}(k-1)! \beta L(\lambda)(\lambda x)^{-k} + o\left(\lambda^{-k} L(\lambda)\right)$$

if and only if there exists a $(k-1)$-primitive $g$ of $f$ satisfying

$$g(\lambda x) = \gamma \lambda^{-1} L(\lambda) \delta(x) + \beta L(\lambda)(\lambda x)^{-1} + o\left(\lambda^{-1} L(\lambda)\right) \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

**Proof.** Apply Proposition 2.17. □
We should introduce some notation that will be needed. In the following for all \( n \in \mathbb{N} \) we denote by \( l_n \) the primitive of \( \log|x| \) with the property that \( l_n(0) = 0 \) and \( l'_n = l_{n-1} \). We have an explicit formula for them:

\[
l_n(x) = \frac{x^n}{n!} \log|x| - \frac{x^n}{n!} \sum_{j=1}^{n} \frac{1}{j}, \quad x \in \mathbb{R},
\]

which can be easily verified by direct differentiation. They satisfy

\[
l_n(ax) = a^n l_n(x) + \frac{(ax)^n}{n!} \log a, \quad a > 0.
\]

(2.72)

**Theorem 2.31.** Let \( f \in \mathcal{D}'(\mathbb{R}) \) have quasi-asymptotics at \( \pm \infty \) (resp. at the origin) of the form

\[
f(\lambda x) = \gamma \lambda^{-1} L(\lambda) \delta(x) + \beta \lambda^{-1} L(\lambda) x^{-1} + o \left( \lambda^{-1} L(\lambda) \right)
\]

in \( \mathcal{D}'(\mathbb{R}) \). (2.73)

Then, there exist an associate asymptotically homogeneous function \( b \) satisfying

\[
b(ax) = b(x) + \beta \log a L(x) + o(L(x)), \quad x \to \infty \text{ (resp. } x \to 0^+) \}
\]

(2.74)

an integer \( m \), and a continuous (resp. continuous near 0) \((m+1)\)-primitive \( F \) of \( f \) such that

\[
F(x) = b(|x|) \frac{x^m}{m!} + \gamma \frac{x^m}{2m!} L(|x|) \text{sgn} x - \beta L(|x|) \frac{x^m}{m!} \sum_{j=1}^{m} \frac{1}{j} + o(L(|x|)) \]

(2.75)

as \( x \to \pm \infty \) (resp. \( x \to 0 \)), in the ordinary sense. Conversely, relation (2.75) implies (2.73).

**Proof.** We will show the assertion only at infinity, the proof at the origin is exactly the same. We shall study, as we have been doing, the coefficients of the integration of (2.73). For each \( n \in \mathbb{N} \), choose an \( n \)-primitive \( F_n \) of \( f \) satisfying \( F'_n = F_{n-1} \). We now proceed to integrate (2.73) once, so we obtain

\[
F_1(\lambda x) = b(\lambda) + \frac{\gamma}{2} L(\lambda) \text{sgn} x + \beta L(\lambda) \log|x| + o(L(\lambda)) \in \mathcal{D}'(\mathbb{R}) \}
\]

(2.76)

Now, using the standard trick of evaluating at \( \phi \in \mathcal{D}(\mathbb{R}) \) with the property \( \int_{-\infty}^{\infty} \phi(x)dx = 1 \), one obtains that

\[
b(\lambda a) + \frac{\gamma}{2} L(\lambda a) \int_{-\infty}^{\infty} \text{sgn} x \phi(x)dx + \beta L(\lambda a) \int_{-\infty}^{\infty} \log|x|\phi(x)dx + o(L(\lambda)) = \langle F_1(\lambda a), \phi(x) \rangle = \frac{1}{a} \langle F_1(\lambda x), \phi \left( \frac{x}{a} \right) \rangle
\]

\[
= b(\lambda) + \frac{\gamma}{2} L(\lambda) \int_{-\infty}^{\infty} \text{sgn} x \phi(x)dx + \beta L(\lambda) \int_{-\infty}^{\infty} \log|x|\phi(x)dx + o(L(\lambda)),
\]
Asymptotic Behavior of Generalized Functions

Let $f \in \mathcal{D}'(\mathbb{R})$. Then $f$ has quasi-asymptotics at $\pm \infty$ (resp. at the origin) of the form (2.73) if and only if there exists an $(m+1)$-primitive $F$ of $f$, continuous (resp. continuous near 0), such that for each $a > 0$,

$$
\lim_{x \to \infty} \frac{m! (a^{-m} F(ax) - (-1)^m F(-x))}{x^m L(x)} = \gamma + \beta \log a \quad \text{(resp. \ limit \ as \ } x \to 0^+) . \tag{2.78}
$$

Proof. The limit (2.78) follows from (2.75), (2.74) and (2.72) by direct computation. For the converse, rewrite (2.78) as

$$
a^{-m} F(ax) - (-1)^m F(-x) = (\gamma + \beta \log a) \frac{x^m}{m!} L(x) + o(x^m L(x)) ,
$$

for each $a > 0$. Set

$$
b(x) = m! x^{-m} F(x) - \left( \frac{\gamma}{2} - \beta \sum_{j=1}^{m} \frac{1}{j} \right) L(x) , \quad x > 0 .
$$
By setting $a = 1$ in (2.78), one sees that for $x < 0$,

$$F(x) = b(|x|) \frac{x^m}{m!} + \gamma L(|x|) \frac{x^{m-1}}{2m!} \text{sgn} x - \beta L(|x|) \frac{x^m}{m!} \sum_{j=1}^{m-1} \frac{1}{j} + o(|x|^m L(|x|)).$$

Since

$$a^{-m} F(ax) - F(x) = \beta \log a \frac{x^m}{m!} L(x) + o(x^m L(x)),$$

it is clear that for each $a > 0$,

$$b(ax) = b(x) + \beta \log a L(x) + o(L(x)). \quad \Box$$

It is remarkable that, initially, no uniform condition on $a$ is assumed in (2.78). However, the proof of Theorem 2.32 forces this relation to hold uniformly for $a$ in compact subsets.

We are now ready to state the general structural theorem for negative integral degrees which now follows directly from Lemma 2.6, Theorem 2.31 and Theorem 2.32.

**Theorem 2.33.** Let $f \in \mathcal{D}'(\mathbb{R})$ and let $k$ be a positive integer. Then $f$ has the quasi-asymptotic behavior in $\mathcal{D}'(\mathbb{R})$ at $\pm \infty$ (resp. at the origin), $f(\lambda x) = \gamma \lambda^{-k} L(\lambda) \delta^{(k-1)}(x) + (-1)^{k-1} (k-1)! \beta L(\lambda)(\lambda x)^{-k} + o(\lambda^{-k} L(\lambda))$ if and only if there exist $m \in \mathbb{N}$, $m \geq k$, an associate asymptotically homogeneous function $b$ of degree $0$ at infinity (resp. at the origin) with respect to $L$ satisfying

$$b(ax) = b(x) + \beta \log a L(x) + o(L(x)) \quad x \to \infty \quad (\text{resp. } x \to 0^+),$$

for each $a > 0$, and an $m$-primitive $F$ of $f$ which is continuous (resp. continuous near 0) and satisfies

$$F(x) = b(|x|) \frac{x^{m-k}}{(m-k)!} + \gamma L(|x|) \frac{x^{m-k}}{2(m-k)!} \text{sgn} x - \beta L(|x|) \frac{x^{m-k}}{(m-k)!} \sum_{j=1}^{m-k} \frac{1}{j} + o(|x|^{m-k} L(|x|))$$

as $x \to \pm \infty$ (resp. $x \to 0$), in the ordinary sense. The last property is equivalent to

$$\lim_{x \to \infty} \frac{(m-k)!}{(m-k)!} \left( a^{k-m} F(ax) - (-1)^{m-k} F(-x) \right) = \gamma + \beta \log a \left( \text{resp. } \lim_{x \to 0^+} \right),$$

for each $a > 0$. 

It should be noticed that in (2.79) is not absolutely necessary to assume that the limit is of the form $\gamma + \beta \log a$. Indeed, we have the following corollary.

**Theorem 2.34.** Let $f \in \mathcal{D}'(\mathbb{R})$. Then $f$ has quasi-asymptotics at infinity (resp. at the origin) of degree $-k$, $k \in \mathbb{N}$, if and only if there exists a continuous (resp. continuous near 0) m-primitive $F$ of $f$, $m \geq k$, such that the following limit exists for each $a > 0$,

$$
\lim_{x \to \infty} \frac{(a^{k-m} F(ax) - (-1)^{m-k} F(-x))}{x^{m-k} L(x)} = I(a) \quad (\text{resp. } \lim_{x \to 0^+} \ldots).
$$

**Proof.** We show that $I(a)$ must be of the form $I(1) + \beta \log a$, for some constant $\beta$. We easily see that $I$ is measurable and satisfies

$$
I(ab) = I(a) + I(b) - I(1),
$$

setting $h(x) = e^{I(x) - I(1)}$, one has that $h$ is positive, measurable and satisfies $h(ab) = h(a)h(b)$, from where it follows [148] that $h(x) = x^\beta$, for some $\beta$, and so $I$ has the desired form. $\square$

**2.11 Quasi-asymptotic extension**

We analyze some problems about the extensions of distributions to other spaces together with their quasi-asymptotic properties, we name this problem quasi-asymptotic extension problem.

Let $f \in \mathcal{F}'$ have quasi-asymptotic behavior in $\mathcal{F}'$, that is,

$$
\langle f(\lambda x), \varphi(x) \rangle \sim \lambda^\alpha L(\lambda) \langle g(x), \varphi(x) \rangle, \quad \forall \varphi \in \mathcal{F}.
$$

Suppose that $f$ belongs also to another space $\mathcal{U}'$ such that $\mathcal{F} \subset \mathcal{U}$ (not necessarily densely contained). For various spaces, we investigate in this section the possibility of extending (2.80) to $\mathcal{U}'$, in the sense of obtaining the asymptotic behavior of $\langle f(\lambda x), \varphi(x) \rangle$, for each $\varphi \in \mathcal{U}$. Sometimes the statement “$f \in \mathcal{U}'$” is also part of the problem. The results of the present section were obtained in [171], [172], [173], [186].

**2.11.1 Quasi-asymptotics at the origin in $\mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$**

In this subsection we conclude the discussion initiated in Remark 2 of 2.8 (cf. Theorem 2.21).
Theorem 2.35. Let \( f \in S'(\mathbb{R}) \). If \( f \) has quasi-asymptotic behavior at 0 in \( D'(\mathbb{R}) \), then \( f \) has the same quasi-asymptotic behavior at 0 in the space \( S'(\mathbb{R}) \).

Proof. Let \( \alpha \) be the degree of the quasi-asymptotic behavior. We shall divide the proof into three cases:

1. \( \alpha \notin \{-1, -2, -3, \ldots\} \),
2. \( \alpha = -1 \),
3. \( \alpha = -2, -3, \ldots \)

Suppose its degree is \( \alpha \notin \{-1, -2, -3, \ldots\} \) and

\[
f(\varepsilon x) = C_- L(\varepsilon) \left( \frac{(\varepsilon x)^{\alpha}}{\Gamma(\alpha + 1)} \right) \left( -1 \right)^m C_- H(-x) + C_+ H(x) \]

as \( \varepsilon \to 0^+ \) in \( D'(\mathbb{R}) \). Then, by using Theorem 2.26 and the fact \( f \in S'(\mathbb{R}) \), we conclude the existence of an integer \( m \), a real number \( \beta \) such that

\[
m > -\alpha, \quad \beta > m + \alpha,
\]

and

\[
F(x) = O\left( |x|^{\beta} \right), \quad |x| \to \infty. \quad (2.81)
\]

We make the usual assumptions over \( L \). Assume (cf. 2.10.1) that \( L \) is positive, defined in \((0, \infty)\) and there exists \( M_1 > 0 \) such that

\[
\frac{L(\varepsilon x)}{L(\varepsilon)} \leq M_1 \max \left\{ x^{-\frac{1}{2}}, x^\frac{1}{2} \right\}, \quad \varepsilon, x \in (0, \infty). \quad (2.82)
\]

Let \( \phi \in S(\mathbb{R}) \), then we can decompose \( \phi = \phi_1 + \phi_2 + \phi_3 \), where \( \text{supp} \phi_1 \subseteq (-\infty, 1] \), \( \text{supp} \phi_2 \) is compact and \( \text{supp} \phi_3 \subseteq [1, \infty) \). Observe that since \( \phi_2 \in D(\mathbb{R}) \) we have that

\[
\langle f(\varepsilon x), \phi_2(x) \rangle = (-1)^m C_- \varepsilon^\alpha L(\varepsilon) \left( \frac{x^\alpha}{\Gamma(\alpha + 1)} , \phi_2(x) \right) 
\]

\[
+ C_+ \varepsilon^\alpha L(\varepsilon) \left( \frac{x^\alpha}{\Gamma(\alpha + 1)} , \phi_2(x) \right) + o(\varepsilon^\alpha L(\varepsilon)), \quad \varepsilon \to 0^+. \quad (2.83)
\]
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So, if we want to show (2.83) for \( \phi \), it is enough to show it for \( \phi_3 \) placed instead of \( \phi_2 \) in the relation because by symmetry it would follow for \( \phi_1 \) and hence for \( \phi \). Set

\[
G(x) = \frac{F(x)}{x^\alpha L(x)}, \quad x > 0.
\]

Then

\[
\lim_{x \to 0^+} G(x) = \frac{C_+}{\Gamma(\alpha + m + 1)}.
\]

(2.84)

Relation (2.85) together with (2.82) show that for \( \epsilon \leq 1 \),

\[
\left| G(\epsilon x) \frac{L(\epsilon x)}{L(\epsilon)} x^{\alpha + m} \phi_3^{(m)}(x) \right| \leq 2M_1M_2x^{3+\frac{\beta + 1}{2}} |\phi_3^{(m)}(x)| H(x - 1).
\]

The right hand side of the last estimate belongs to \( L^1(\mathbb{R}) \) and thus we can use the Lebesgue dominated convergence theorem to obtain,

\[
\lim_{\epsilon \to 0^+} \frac{1}{\epsilon^\alpha L(\epsilon)} \langle f(\epsilon x), \phi_3(x) \rangle = \langle x^{\alpha} \frac{C_+}{\Gamma(\alpha + m + 1)}, \phi_3(x) \rangle.
\]

(2.86)
2. Quasi-asymptotics in $F'$

But $g \in S'(\mathbb{R})$, then since the degree of the quasi-asymptotics is 0, the first case implies that (2.86) is valid in $S'(\mathbb{R})$. Therefore

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon}{L(\varepsilon)} \left( f(\varepsilon x), \phi(x) \right) = \lim_{\varepsilon \to 0^+} \frac{1}{L(\varepsilon)} \left( g(\varepsilon x), \frac{\phi(x)}{x} \right) = \beta \int_{1}^{\infty} \frac{\phi(x)}{x} \, dx.$$ 

This shows the case $\alpha = -1$.

It remains to show the theorem when $\alpha \in \{-2, -3, \ldots\}$. Suppose the degree is $-k$, $k \in \{2, 3, \ldots\}$. It is easy to see that any primitive of degree $(k - 1)$ of $f$ has quasi-asymptotics of degree $-1$ at the origin with respect to $L$ (in fact this is the content of Proposition 2.17 when combined with Theorem 2.23). The $(k - 1)$-primitives of $f$ are in $S'(\mathbb{R})$, so we can apply the case $\alpha = -1$ to them, and then, by differentiation, it follows that $f$ has quasi-asymptotics at the origin in $S'(\mathbb{R})$.

This completes the proof of Theorem 2.35. \[\square\]

2.11.2 Quasi-asymptotic extension problem in $D'(0, \infty)$

The purpose of this subsection is to study the following problem. Suppose that a distribution $f \in D'(\mathbb{R})$ with support in $[0, \infty)$ has quasi-asymptotics of degree $\alpha$ in the space $D'(0, \infty)$, that is, for each $\phi \in D(0, \infty)$

$$\left( f(\lambda x), \phi(x) \right) \sim \lambda^{\alpha} L(\lambda) \left( g(x), \phi(x) \right). \quad (2.87)$$

What can we say about the quasi-asymptotic properties of $f$ in $D'(\mathbb{R})$?

We can apply the techniques from 2.10.3 and 2.10.5 to give a complete answer to this question. The answer depends on $\alpha$. We start with the quasi-asymptotic behavior at infinity, the same arguments are applicable to quasi-asymptotics at the origin.

Let us start with the case $\alpha > -1$. It is not difficult to show that $g$ must be of the form $C x^{\alpha}_+ / \Gamma(\alpha + 1)$, for some constant $C$. Next, Proposition 2.17 still holds replacing the space $D'(\mathbb{R})$ by $D'(0, \infty)$ (actually this holds without the restriction $\alpha > -1$). Hence, the same argument given in Theorem 2.26 applies here, but this time we only require the uniform convergence on $[1/2, 2]$, and hence we can still conclude the existence of the integer such that (2.61) holds with the limit taken only as $x \to \infty$. Actually, because $\alpha > -1$, relation (2.61) holds for any $m$-primitive of $f$. Let $F$ be the $m$-primitive of $f$ supported on the interval $[0, \infty)$, then we have that

$$F(x) \sim \frac{C x^{\alpha+m} L(x)}{\Gamma(\alpha + m + 1)}, \quad x \to \infty,$$
so we have that \( F(\lambda x) = C L(\lambda) (\lambda x)^{\alpha + m} / \Gamma(\alpha + m + 1) + o(\lambda^{\alpha + m} L(\lambda)) \) in the space \( \mathcal{S}'(\mathbb{R}) \), differentiating \( m \)-times, we obtain the following result.

**Theorem 2.36.** Let \( f \in \mathcal{D}'(\mathbb{R}) \) be supported on \([0, \infty)\). If \( f \) has quasi-asymptotic behavior of degree \( \alpha > -1 \) in \( \mathcal{D}'(0, \infty) \), then it is a tempered distribution and has the same quasi-asymptotic behavior in the space \( \mathcal{S}'(\mathbb{R}) \).

Suppose now that \( \alpha < -1 \) and \( \alpha \) is not a negative integer. This case differs from the last one essentially in one point, we cannot conclude (2.61) for every \( m \)-primitive of \( f \) but only for some of them. In any case, denoting again by \( F \) the \( m \)-primitive (we keep \( m > -\alpha - 1 \)) of \( f \) supported on \([0, \infty)\), we have that there exists a polynomial of degree at most \( m - 1 \) such that

\[
F(x) - p(x) \sim \frac{C x^{\alpha + m} L(x)}{\Gamma(\alpha + m + 1)}, \quad x \to \infty;
\]

therefore,

\[
F(\lambda x) = \frac{C L(\lambda) (\lambda x)^{\alpha + m}}{\Gamma(\alpha + m + 1)} + \sum_{j=0}^{m-1} a_j (\lambda x)_+^j + o(\lambda^{\alpha + m} L(\lambda)) \quad \text{as } \lambda \to \infty,
\]

in the space \( \mathcal{S}'(\mathbb{R}) \), for some constants \( a_0, \ldots, a_{m-1} \). Thus, our arguments immediately yield the next theorem.

**Theorem 2.37.** Let \( f \in \mathcal{D}'(\mathbb{R}) \) be supported on \([0, \infty)\). Suppose that

\[
f(\lambda x) = C L(\lambda) \left( \frac{\lambda x}{\Gamma(\alpha + 1)} \right)^{\alpha} + o(\lambda^{\alpha} L(\lambda)) \quad \text{as } \lambda \to \infty \quad \text{in } \mathcal{D}'(0, \infty).
\]

If \( \alpha < -1 \) and \( \alpha \) is not a negative integer, then \( f \) is a tempered distribution. Moreover, there exist constants \( a_0, a_1, \ldots, a_n \) \((n < -\alpha - 1)\) such that

\[
f(\lambda x) = C L(\lambda) \left( \frac{\lambda x}{\Gamma(\alpha + 1)} \right)^{\alpha} + \sum_{j=0}^{n} a_j \frac{\delta^{(j)}(x)}{\lambda^j + 1} + o(\lambda^{\alpha} L(\lambda)) \quad \text{as } \lambda \to \infty \quad \text{in } \mathcal{S}'(\mathbb{R}).
\]

When \( \alpha = -k \), \( k \) being a positive integer, the distribution \( g \) in (2.87) must have the form \( C x^{-k} \in \mathcal{D}'(0, \infty) \), for some constant \( C \); these distributions are homogeneous as elements of \( \mathcal{D}'(0, \infty) \), but they do not have homogeneous extensions to \( \mathcal{D}(\mathbb{R}) \). The behavior of \( f(\lambda x) \) as \( \lambda \to \infty \) in \( \mathcal{S}'(\mathbb{R}) \) is described in the next theorem.

We denote by \( \text{Pf}(H(x)/x^k) \) the distribution

\[
\left\langle \text{Pf} \left( \frac{H(x)}{x^k} \right), \phi(x) \right\rangle = \text{F.p.} \int_0^\infty \frac{\phi(x)}{x^k} dx, \quad \phi \in \mathcal{D}(\mathbb{R}),
\]
where F.p. stands for the Hadamard finite part of the divergent integral [56].

**Theorem 2.38.** Let \( f \in \mathcal{D}'(\mathbb{R}) \) be supported on \([0, \infty)\). Suppose that

\[
f(\lambda x) = CL(\lambda) \frac{H(x)}{(\lambda x)^k} + o \left( \frac{L(\lambda)}{\lambda^k} \right) \quad \text{as} \; \lambda \to \infty \; \text{in} \; \mathcal{D}'(0, \infty),
\]

where \( k \) is a positive integer. Then \( f \) is a tempered distribution and there exist an associate asymptotically homogeneous function \( b \) satisfying

\[
b(ax) = b(x) + \frac{(-1)^{k-1}}{(k-1)!} CL(x) \log a + o(L(x)), \quad x \to \infty,
\]

for each \( a > 0 \), and constants \( a_0, a_1, \ldots, a_{k-2} \) such that

\[
f(\lambda x) = C \frac{L(\lambda)}{\lambda^k} Pf \left( \frac{H(x)}{x^k} \right) + \frac{b(\lambda)}{\lambda^k} \delta^{(k-1)}(x) + \sum_{j=0}^{k-2} a_j \frac{\delta^{(j)}(x)}{\lambda^{j+1}} + o \left( \frac{L(\lambda)}{\lambda^k} \right)
\]

as \( \lambda \to \infty \) in \( \mathcal{S}'(\mathbb{R}) \).

**Proof.** For each \( n \in \mathbb{N} \), let \( F_n \) denote the \( n \)-primitive of \( f \) with support in \([0, \infty)\). Set \( C_1 = (-1)^{k-1} C'/(k-1)! \). Adapting the arguments of 2.10.5 and reasoning as in the previous two cases, we obtain the existence of a positive integer \( m > k \) such that \( F_m \) is continuous and

\[
F_m(x) = b_1(x) \frac{x^{m-k}}{(m-k)!} - C_1 L(x) \frac{x^{m-k}}{(m-k)!} \sum_{j=1}^{m-k} \frac{1}{j} \frac{p_{m-1}(x) + o(x^{m-k} L(x))}{H(x)}
\]

as \( x \to \infty \), where \( b_1 \) is a locally integrable associate asymptotically homogeneous function satisfying (2.88) and \( p_{m-1} \) is a polynomial of degree at most \( m-1 \). Throwing away the irrelevant terms of the polynomial \( p_{m-1} \) and using Theorem 2.27, we obtain the following asymptotic expansion as \( \lambda \to \infty \) in the space \( \mathcal{S}'(\mathbb{R}) \),

\[
F_m(\lambda x) = b_1(\lambda) \frac{(\lambda x)^{m-k}}{(m-k)!} + C_1 \lambda^{m-k} L(\lambda) l_{m-k}(x) H(x)
\]

\[
\quad + \sum_{j=0}^{k-2} a_j \frac{(\lambda x)^{m-j-1}}{(m-j-1)!} + o(\lambda^{m-k} L(\lambda)).
\]

Differentiating \((m - k)\)-times this expansion, we have that

\[
F_k(\lambda x) = b_1(\lambda) H(x) + C_1 L(\lambda) H(x) \log x + \sum_{j=0}^{k-2} a_j \frac{(\lambda x)^{k-j-1}}{(k-j-1)!} + o(L(\lambda)).
\]

(2.90)
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The well known formulas [56], p. 68,
\[ \frac{d}{dx}(H(x) \log x) = \text{Pf}\left( \frac{H(x)}{x} \right) \]
and
\[ \frac{d}{dx}\left( \text{Pf}\left( \frac{H(x)}{x^n} \right) \right) = -n\text{Pf}\left( \frac{H(x)}{x^{n+1}} \right) + \frac{(-1)^n\delta^{(n)}(x)}{n!} \]
imply that
\[ \frac{d^{k-1}}{dx^{k-1}} \left( \text{Pf}\left( \frac{H(x)}{x^n} \right) \right) = (-1)^{k-1}(k-1)!\text{Pf}\left( \frac{H(x)}{x^k} \right) - \delta^{(k-1)}(x) \sum_{j=1}^{k-1} \frac{1}{j}. \]

Hence, differentiating \( (2.90) \) \( k \)-times, one has \( (2.89) \) with
\[ \frac{d^k}{dx^k}(f(x)) = (-1)^k \mathcal{C} \left( \sum_{j=1}^{k-1} \frac{1}{j} \right) \mathcal{L}(x). \]

The corresponding result at the origin is stated in the next theorem.

\textbf{Theorem 2.39.} Let \( f_0 \in \mathcal{D}'(0, \infty) \). Let \( L \) be slowly varying at the origin and \( \alpha \in \mathbb{R} \). Suppose that
\[ f_0(\varepsilon x) = e^{\alpha}L(\varepsilon)g_0(x) + o(\varepsilon^\alpha L(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{in} \quad \mathcal{D}'(0, \infty), \quad (2.91) \]
g_0 \in \mathcal{D}'(0, \infty). Then \( f_0 \) admits extensions to \([0, \infty)\). Let \( f \) be any of such extensions. Then \( f \) has the following asymptotic properties at the origin:

(i) If \( \alpha \notin -\mathbb{N} \), then there exist constants \( C, a_0, \ldots, a_n \) such that
\[ f(\varepsilon x) = \sum_{j=0}^{n} a_j \delta^{(j)}(\varepsilon x) + Ce^{\alpha}L(\varepsilon)x^\alpha + o(\varepsilon^\alpha L(\varepsilon)), \]
as \( \varepsilon \to 0^+ \) in \( \mathcal{D}'(\mathbb{R}) \).

(ii) If \( \alpha = -k, k \in \mathbb{N} \), then there are constants \( C, a_0, \ldots, a_n \) and an associate asymptotically homogeneous function of degree 0 with respect to \( L \) satisfying
\[ b(ax) = b(x) + \frac{(-1)^{k-1}}{(k-1)!}CL(x) \log a + o(L(x)), \quad (2.92) \]
such that
\[ f(\varepsilon x) = \sum_{j=0}^{n} a_j \delta^{(j)}(\varepsilon x) + b(\varepsilon)\delta^{(k-1)}(\varepsilon x) + Ce^{-k}L(\varepsilon)\text{Pf}\left( \frac{H(x)}{x^k} \right) + o(e^{-k}L(\varepsilon)), \quad (2.93) \]
as \( \varepsilon \to 0^+ \) in \( \mathcal{D}'(\mathbb{R}) \).
2.11.3 Quasi-asymptotics at infinity and spaces $V'_\beta(\mathbb{R})$

Sometimes it is very useful to have the right of evaluating (2.41) in more test functions than in $\mathcal{S}(\mathbb{R})$, this section is dedicated to give some conditions under the test function which guarantee one can do this for the quasi-asymptotic behavior at $\pm\infty$. We shall now improve Theorem 2.10 and its corollary (cf. 2.6).

We need the following definition.

**Definition 2.13.** Let $\phi \in \mathcal{E}(\mathbb{R})$ and $\beta \in \mathbb{R}$. We say that

$$
\phi(x) = O(|x|^\beta) \text{ strongly as } |x| \to \infty,
$$

if for each $m \in \{0, 1, 2, \ldots\}$

$$
\phi^{(m)}(x) = O(|x|^\beta - m) \text{ as } |x| \to \infty.
$$

The set of $\phi$ satisfying Definition 2.13 for a particular $\beta$ forms the space $V'_{\beta}(\mathbb{R})$ which we topologize in the obvious way [56]. These spaces and their dual spaces are very important in the theory of asymptotic expansions of distributions [56]. In fact, if we set $\mathcal{V}(\mathbb{R}) = \bigcup V'_{\beta}(\mathbb{R})$ (the union having a topological meaning), we have that $V'(\mathbb{R})$ is the space of distributional small distributions at infinity [47], [56], they satisfy the moment asymptotic expansion at infinity (cf. Example 3 in 2.9). We point out that in [56] these spaces are denoted by $V'_\beta = K'_\beta$ and $V' = K'$.

The next theorem shows that if $f$ has quasi-asymptotics at $\pm\infty$, then the distributional evaluation of $f$ at $\phi \in V'_{\beta}(\mathbb{R})$ makes sense under some conditions on $\beta$, specifically, we show that $f$ has extensions to some of the spaces $V'_{\beta}(\mathbb{R})$.

**Theorem 2.40.** Let $f \in \mathcal{D}'(\mathbb{R})$ have quasi-asymptotic behavior of degree $\alpha$ at $\pm\infty$ with respect to the slowly varying function $L$. If $\alpha + \beta < -1$, then $f$ admits extensions to $V'_{\beta}(\mathbb{R})$.

**Proof.** Let $\sigma > 0$ such that $\alpha + \beta + \sigma < -1$, then from Theorem 2.26, Theorem 2.33 and Lemma 2.5 we deduce that there exist $m \in \mathbb{N}$ and a continuous $m$-primitive of $f$, say $F$, such that

$$
F(x) = O(|x|^{m+\alpha+\sigma}) \text{ as } |x| \to \infty.
$$

(2.96)
Notice that here we have used that $L(x) = O(x^{\sigma})$ as $x \to \infty$ [148]. So it is evident that the extension of $f$ to $V_\beta(\mathbb{R})$ is given by

$$\langle f(x), \phi(x) \rangle = (-1)^m \int_{-\infty}^{\infty} F(x) \phi^{(m)}(x) dx, \quad \phi \in V_\beta(\mathbb{R}),$$

which in view of (2.95) and (2.96) is well-defined and defines an element of $V'_\beta(\mathbb{R})$. □

We now show that the quasi-asymptotic behavior remains valid in $V'_\beta(\mathbb{R})$ for one extension of $f$, with the assumption under $\beta$ imposed in Theorem 2.40.

**Theorem 2.41.** Let $f \in D'(\mathbb{R})$ have quasi-asymptotic behavior at $\infty$ of degree $\alpha$ with respect to a slowly varying function $L$, then $f$ admits an extension to $V_\beta(\mathbb{R})$ which has the same quasi-asymptotics in $V'_\beta(\mathbb{R})$, provided that $\alpha + \beta < -1$.

**Proof.** The proof is similar to that of Theorem 2.35 with some modifications in the estimates. We use one of the extensions of $f$ found in Theorem 2.40; denote the extension by $\tilde{f}$. We shall divide the proof into two cases: $\alpha \notin \{-1, -2, -3, \ldots\}$ and $\alpha \in \{-1, -2, -3, \ldots\}$.

Suppose its degree is $\alpha \notin \{-1, -2, -3, \ldots\}$ and

$$f(\lambda x) = C_- L(\lambda) \frac{(\lambda x)^\alpha}{\Gamma(\alpha + 1)} + C_+ L(\lambda) \frac{(\lambda x)^\alpha}{\Gamma(\alpha + 1)} + o(\lambda^\alpha L(\lambda)) \text{ as } \lambda \to \infty,$$

in $D'(\mathbb{R})$. Find $\sigma > 0$ such that $\alpha + \beta + \sigma < -1$. Then from Theorem 2.26, there are an $m$ such that $m + \alpha > 0$ and a continuous $m$-primitive $F$ of $f$ such that

$$F(x) = \frac{x^m |x|^\alpha L(|x|) (C_- H(-x) + C_+ H(x)) + o \left(|x|^{m+\alpha} L(|x|)\right)}{\Gamma(m + \alpha + 1)}, \quad x \to \infty.$$  

We recall that $H$ denotes the Heaviside function. We make the usual assumptions over $L$ (cf. 2.10.1), assume that $L$ is positive, defined and continuous in $(0, \infty)$ and there exists $M_1 > 0$ such that

$$\frac{L(\lambda x)}{L(\lambda)} \leq M_1 \max \{x^\sigma, x^{-\sigma}\}, \quad \lambda \geq 1, \quad x \in (0, \infty). \quad (2.98)$$

Let $\phi \in V_\beta(\mathbb{R})$. As in the proof of Theorem 2.35, we may assume that $\text{supp } \phi \subseteq [1, \infty)$, and the proof would be complete after we show

$$\langle \tilde{f}(\lambda x), \phi(x) \rangle \sim C_+ \lambda^\alpha L(\lambda) \left(\frac{x^\alpha}{\Gamma(\alpha + 1)}\right) \phi(x),$$

where $L(\lambda) = o(x^\alpha)$ as $\lambda \to \infty$. □
2. Quasi-asymptotics in $\mathcal{F}'$

as $\lambda \to \infty$.

Set

$$G(x) = \frac{F(x)}{x^{\alpha+m}L(x)} \quad \text{for} \quad x \geq 1,$$

(2.100)

then

$$\lim_{x \to \infty} G(x) = \frac{C_+}{\Gamma(\alpha + m + 1)}.$$  (2.101)

So, we can find a constant $M_2 > 0$ such that

$$|G(x)| < M_2, \quad \text{globally.}$$

Relation (2.102) together with (2.98) show that for $\lambda \geq 1$,

$$\left| G(\lambda x) \frac{L(\lambda x)}{L(\lambda)} x^{\alpha+m} \phi^{(m)}(x) \right| \leq M_1 M_2 x^{\alpha+m-\sigma} \left| \phi^{(m)}(x) \right| H(x-1).$$

Since $\phi \in \mathcal{V}_\beta(\mathbb{R})$, the right hand side of the last estimate belongs to $L^1(\mathbb{R})$ and thus we can use the Lebesgue dominated convergence theorem to obtain,

$$\lim_{\lambda \to \infty} \frac{\langle \tilde{f}(\lambda x), \phi(x) \rangle}{\lambda^\alpha L(\lambda)} = \lim_{\lambda \to \infty} (-1)^m \int_0^\infty G(\lambda x) \frac{L(\lambda x)}{L(\lambda)} x^{\alpha+m} \phi^{(m)}(x) dx$$

$$= (-1)^m \frac{C_+}{\Gamma(\alpha + m + 1)} \int_0^\infty x^{\alpha+m} \phi^{(m)}(x) dx$$

$$= C_+ \left\langle \frac{x^\alpha}{\Gamma(\alpha + 1)}, \phi(x) \right\rangle.$$

This shows the result in the case $\alpha \notin \{-1,-2,-3,\ldots\}$.

We now consider the case $\alpha = -k$, $k \in \mathbb{N}$. Assume that

$$f(\lambda x) = \gamma \lambda^{-k} L(\lambda) \delta^{(k-1)}(x) + \beta \lambda^{-k} L(\lambda) x^{-k} + o(\lambda^{-k} L(\lambda)),$$

as $\lambda \to \infty$ in $\mathcal{D}'(\mathbb{R})$. As in the last case, it suffices to assume that $\phi \in \mathcal{V}_\beta(\mathbb{R})$, supp $\phi \subseteq [1, \infty)$ and show that

$$\lim_{\lambda \to \infty} \frac{\lambda^k}{L(\lambda)} \left\langle \tilde{f}(\lambda x), \phi(x) \right\rangle = \beta \int_1^\infty \frac{\phi(x)}{x^k} \, dx.$$

So, set $g(x) = x^k f(x)$, then

$$g(\lambda x) = \beta L(\lambda) + o(L(\lambda)) \quad \text{as} \quad \lambda \to \infty \quad \text{in} \quad \mathcal{D}'(\mathbb{R}).$$

(2.103)

But $\phi \in \mathcal{V}_\beta(\mathbb{R})$ implies $\phi(x)/x^k \in \mathcal{V}_{\beta-k}(\mathbb{R})$ then since the degree of the quasi-asymptotic behavior of $g$ is 0, last case implies that (2.103) is valid in $\mathcal{V}_{\beta-k}(\mathbb{R})$ for a suitable extension $\tilde{g}$ because $\beta - k < -1$, therefore

$$\lim_{\lambda \to \infty} \frac{\lambda^k}{L(\lambda)} \left\langle \tilde{f}(\lambda x), \phi(x) \right\rangle = \lim_{\lambda \to \infty} \frac{1}{L(\lambda)} \left\langle \tilde{g}(\lambda x), \frac{\phi(x)}{x^k} \right\rangle = \beta \int_1^\infty \frac{\phi(x)}{x^k} \, dx.$$
This completes the proof of Theorem 2.41.

The importance Theorem 2.41 lies in the fact that we can relax the growth restrictions over the test functions, this permits to apply quasi-asymptotics to obtain ordinary asymptotics in many interesting situations, for example for certain integral transforms or for solutions to partial differential equations. We discuss a simple example.

Example. Let \( f \in \mathcal{D}'(\mathbb{R}) \) have quasi-asymptotic behavior at infinity of degree \( \alpha < 1 \),
\[
f(\lambda x) = \lambda^{\alpha} L(\lambda) g(x) + o(\lambda^{\alpha} L(\lambda)) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}).
\]

Consider the Poisson kernel,
\[
P(t) = \frac{1}{\pi (t^2 + 1)}.
\]
Clearly \( P \in \mathcal{V}_2(\mathbb{R}) \). By Theorem 2.41, \( f \) has an extension \( \tilde{f} \) such that the evaluation of \( \tilde{f} \) at \( P \) is well defined and \( \tilde{f} \) preserves the quasi-asymptotic properties of \( f \). Thus
\[
U(z) = U(x + yi) = \left\langle f(t), \frac{1}{y} P \left( \frac{x - t}{y} \right) \right\rangle
\]
is a solution of the boundary value problem
\[
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad U(x + i0^+) = f(x) \quad \text{in } \mathcal{D}'(\mathbb{R}).
\]
Using Theorem 2.41, we can find the asymptotic behavior of \( U \) at infinity over cones. Indeed, let \( 0 < \sigma < \pi/2 \), then Theorem 2.41 implies that as \( r \to \infty \)
\[
U(re^{i\theta}) \sim \sin^{\alpha}(\theta) C_\theta r^{\alpha} L(r), \quad \text{uniformly for } \theta \in [\sigma, \pi - \sigma],
\]
where \( C_\theta = g * P(\cot \theta) \).

### 2.12 Quasi-asymptotic boundedness

This section is intended to study the structure of the distributional relation
\[
f(\lambda x) = O(\rho(\lambda)),
\]
where here \( \lambda \to \infty \) or \( \lambda \to 0^+ \) and \( \rho \) is a regularly varying function. Our approach to the problem follows the exposition from [172]. Distributions satisfying this relation will be called quasi-asymptotically bounded distributions, we make this more precise in the following definition.
2. Quasi-asymptotics in $\mathcal{F}'$

**Definition 2.14.** Let $L$ be a slowly varying function at infinity (respectively at the origin). We say that $f \in \mathcal{D}'$ is quasi-asymptotically bounded at infinity (at the origin) in $\mathcal{D}'(\mathbb{R})$ with respect to $\lambda^\alpha L(\lambda)$, $\alpha \in \mathbb{R}$, if

$$
(f(\lambda x), \phi(x)) = O(\lambda^\alpha L(\lambda)) \quad \text{as} \quad \lambda \to \infty \quad \forall \phi \in \mathcal{D}(
abla), \quad (2.104)
$$

(respectively $\lambda \to 0^+$). If (2.104) holds, it is also said that $f$ is quasi-asymptotically bounded of degree $\alpha$ at infinity (at the origin) with respect to the slowly varying function $L$. We express (2.104) by

$$
f(\lambda x) = O(\lambda^\alpha L(\lambda)) \quad \text{as} \quad \lambda \to \infty \quad \text{in} \quad \mathcal{D}'(\mathbb{R}), \quad (2.105)
$$

(respectively $\lambda \to 0^+$).

Note that in analogy to the quasi-asymptotic behavior of distributions, we may talk about (2.105) in other spaces of distributions. In the case at infinity, It will follow from our structural theorem that $f \in \mathcal{S}'(\mathbb{R})$ and actually the relation (2.105) holds in $\mathcal{S}'(\mathbb{R})$. The case at the origin is related to the problem of extension of distributions from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$. Indeed, if $f \in \mathcal{D}'(\mathbb{R} \setminus \{0\})$ and (2.104) holds for all $\phi \in \mathcal{D}(\mathbb{R} \setminus \{0\})$, we will see later that $f$ admits an extension to $\mathbb{R}$.

We now proceed to obtain the structure of quasi-asymptotically bounded distributions. For this aim, the program established in 2.10 will be followed. We will integrate the relation (2.105) and the coefficients of this integration will satisfy the properties of the following definition.

**Definition 2.15.** A function $b$ is said to be asymptotically homogeneously bounded of degree $\alpha$ at infinity with respect to the slowly varying function $L$ if it is measurable and defined in some interval $(A, \infty)$, $A > 0$, and for each $a > 0$

$$
b(ax) = a^\alpha b(x) + O(L(x)), \quad x \to \infty. \quad (2.106)
$$

Similarly, one defines asymptotically homogeneously bounded functions at the origin. Our first goal is to study the asymptotic properties of this class of functions. Proceeding as in Lemma 2.2, or using the results of [148], Section 2.4, one has that (2.106) must hold uniformly in compact subsets of $(0, \infty)$. Most of the proofs of the following results are the analog to those for asymptotically homogeneous functions by replacing the $o$ symbol by the $O$ symbol and making obvious modifications to the estimates, therefore they will be omitted.
Proposition 2.19. Let $b$ be asymptotically homogeneously bounded at infinity (at the origin) with respect to the slowly varying function $L$. If the degree is negative (respectively positive), then $b(x) = O(L(x))$, as $x \to \infty$ ($x \to 0^+$).

Proposition 2.20. Let $b$ be asymptotically homogeneously bounded at infinity (at the origin) with respect to the slowly varying function $L$. If the degree $\alpha$ is positive (respectively negative), then there exists a constant $\gamma$ such that $b(x) = \gamma x^\alpha + O(L(x))$, as $x \to \infty$ ($x \to 0^+$).

Note that for the case at infinity since $L(x) = O(x^\sigma)$ as $x \to \infty$, for any $\sigma > 0$, then any asymptotically homogeneously bounded function of degree 0 at infinity satisfies that $b(x)/x^\sigma$ is asymptotically homogeneously bounded of degree $-\sigma$ with respect to the trivial slowly varying function $L \equiv 1$ and hence by Proposition 2.19 it satisfies $b(x) = O(x^\sigma)$ as $x \to \infty$, hence for large argument it is a regular tempered distribution. Similarly, any asymptotically homogeneously bounded function of degree 0 at the origin satisfies $b(x) = O(x^{-\sigma})$ as $x \to 0^+$, for any $\sigma > 0$, consequently it is a distribution for small argument. The proof of the next proposition is totally analogous to those of Theorems 2.24 and 2.27, and therefore it will be omitted.

Proposition 2.21. Let $b$ be asymptotically homogeneously bounded of degree zero at the infinity (at the origin) with respect to the slowly varying function $L$. Suppose that $b$ is locally integrable on $[A, \infty)$ (respectively $(0, A]$). Then

$$b(\lambda x)H(\lambda x - A) = b(\lambda)H(x) + O(L(\lambda)) \quad \text{as } \lambda \to \infty \quad \text{in } \mathcal{S}'(\mathbb{R}), \quad (2.107)$$

(resp. $b(\lambda x)(H(x) - H(\lambda x - A)) = b(\lambda)H(x) + O(L(\lambda))$ as $\lambda \to 0^+$ in $\mathcal{D}'(\mathbb{R})$).

Corollary 2.3. Let $b$ be an asymptotically homogeneously bounded function of degree 0 at infinity (at the origin) with respect to $L$. Then, there exists $c \in C^\infty[0, \infty)$, being asymptotically homogeneously bounded of degree 0, such that $b(x) = c(x) + O(L(x))$ as $x \to \infty$ (resp. as $x \to 0^+$).

Proof. We only show the assertion at infinity, the case at the origin is similar. Find $B$ such that $b$ is locally bounded in $[B, \infty)$. Take $\phi \in \mathcal{D}(\mathbb{R})$ supported in $(0, \infty)$ such that $\int_0^\infty \phi(t)dt = 1$ and set $c(x) = \int_{B/x}^\infty b(xt)\phi(t)dt$, the corollary now follows from Proposition 2.21. □
The main connection between quasi-asymptotically bounded distributions and asymptotically homogeneously bounded functions is given in the next proposition, again the proof will be omitted since it is analogous to that of Proposition 2.17.

**Proposition 2.22.** Let \( f \in D'(\mathbb{R}) \) be quasi-asymptotically bounded of degree \( \alpha \) at infinity (at the origin) with respect to the slowly varying function \( L \). Let \( m \in \mathbb{N} \). Then, for any given \( F_m \), an \( m \)-primitive of \( f \) in \( D'(\mathbb{R}) \), there exist functions \( b_0, \ldots, b_{m-1} \), continuous on \((0, \infty)\), such that

\[
F_m(\lambda x) = \sum_{j=0}^{m-1} \lambda^\alpha b_j(\lambda) \frac{x^{m-1-j}}{(m-1-j)!} + O\left(\lambda^{\alpha+m} L(\lambda)\right) \quad \text{in} \ D'(\mathbb{R}),
\]

as \( \lambda \to \infty \) (respectively \( \lambda \to 0^+ \)), where each \( b_j \) is asymptotically homogeneously bounded of degree \(-\alpha - j - 1\) with respect to \( L \).

Thus we obtain from Propositions 2.19–2.22 our first structural theorem.

**Theorem 2.42.** Let \( f \in D'(\mathbb{R}) \) and \( \alpha \notin -\mathbb{N} \). Then \( f \) is quasi-asymptotically bounded of degree \( \alpha \) at infinity (at the origin) with respect to the slowly varying function \( L \) if and only if there exist \( m \in \mathbb{N}, m + \alpha > 0 \), and a continuous (continuous near 0) \( m \)-primitive \( F \) of \( f \) such that

\[
F(x) = O\left(|x|^{m+\alpha} L(|x|)\right),
\]

as \( |x| \to \infty \) (respectively \( x \to 0 \)) in the ordinary sense. Moreover, in the case at infinity, \( f \) belongs to \( S'(\mathbb{R}) \) and is quasi-asymptotically bounded of degree \( \alpha \) with respect to \( L \) in \( S'(\mathbb{R}) \).

**Proof.** We only discuss the case at infinity, the proof of the assertion at the origin is similar to this case. It follows from Proposition 2.22, Proposition 2.19 and Proposition 2.20 that given \( m \in \mathbb{N} \) and an \( m \)-primitive \( F_m \), there is a polynomial \( p_{m-1} \) of degree at most \( m - 1 \) such that

\[
F_m(\lambda x) = p_{m-1}(\lambda x) + O(\lambda^{\alpha+m} L(\lambda)) \quad \text{as} \ \lambda \to \infty \quad \text{in} \ D'(\mathbb{R}),
\]

from the definition of boundedness in \( D'(\mathbb{R}) \) it follows that there is an \( m > -\alpha \) such that (2.110) holds uniformly for \( x \in [-1, 1] \). We let \( F = F_m - p_{m-1} \), so by taking \( x = -1, x = 1 \) and replacing \( \lambda \) by \( x \) in (2.110) we obtain (2.109). The converse follows by observing that (2.109) implies that \( F(x) = O(\lambda^{\alpha+m} L(\lambda)) \) in \( S'(\mathbb{R}) \) which gives the result after differentiating \( m \)-times. \( \square \)
We now analyze the case of negative integral degree.

**Theorem 2.43.** Let \( f \in \mathcal{D}'(\mathbb{R}) \) and let \( k \) be a positive integer. Then \( f \) is quasi-asymptotically bounded of degree \(-k\) at infinity (at the origin) with respect to \( L \) if and only if there exist \( m > k \), an asymptotically homogeneously bounded function \( b \) of degree 0 at infinity (at the origin) with respect to \( L \) and a continuous (continuous near 0) \( m \)-primitive \( F \) of \( f \) such that

\[
F(x) = b(|x|) x^{m-k} + O \left( |x|^{m-k} L(|x|) \right),
\]

as \(|x| \to \infty \) (\( x \to 0 \)). Moreover \((2.111)\) is equivalent to have

\[
a^{k-m} F(ax) - (-1)^{m-k} F(-x) = O \left( x^{m-k} L(x) \right),
\]

as \( x \to \infty \) (\( x \to 0^+ \)), for each \( a > 0 \). In the case at infinity, it follows that \( f \) is quasi-asymptotically bounded of degree \(-k\) with respect to \( L \) in \( \mathcal{S}'(\mathbb{R}) \).

**Proof.** Again we only give the proof of the assertion at infinity, the case at the origin is similar. If \( f(\lambda x) = O(\lambda^{-k} L(\lambda)) \) in \( \mathcal{D}'(\mathbb{R}) \), then after \( k-1 \) integrations Proposition 2.22 and Proposition 2.20 provide us of a \((k-1)\)-primitive of \( f \) which is quasi-asymptotically bounded of degree \(-1\) at infinity with respect to \( L \), hence we may assume that \( k = 1 \). Next, Proposition 2.22, Proposition 2.19 and the definition of boundedness in \( \mathcal{D}'(\mathbb{R}) \) give to us the existence of an \( m > 1 \), an asymptotically homogeneously bounded function of degree \(-1\) with respect to \( L \) and an \( m \)-primitive \( F \) of \( f \) such that \( F(\lambda x) \) is continuous for \( x \in [-1,1] \) (hence \( F \) is continuous on \( \mathbb{R} \)) and \( F(\lambda x) = \lambda^{m-1} b(\lambda) x^{m-1} + O(\lambda^{m-1} L(\lambda)) \) as \( \lambda \to \infty \) uniformly for \( x \in [-1,1] \). By taking \( x = -1, x = 1 \) and replacing \( \lambda \) by \( x \) one gets \((2.111)\).

Assume now \((2.111)\), by using Corollary 2.3, we may assume that \( b \) is locally integrable on \([0,\infty)\). This allows the application of Proposition 2.21 to deduce that \( F(\lambda x) = \lambda^{m-1} b(\lambda) x^{m-1} + O(\lambda^{m-1} L(\lambda)) \) as \( \lambda \to \infty \) in \( \mathcal{S}'(\mathbb{R}) \) and hence the converse follows by differentiating \( m \)-times. That \((2.111)\) implies \((2.112)\) is a simple calculation; conversely, setting \( b(x) = x^{k-m} F(x) \) for \( x > 0 \), one obtains \((2.111)\). \( \square \)

**Remark 1.1.** Even if not assumed initially, the proof of Theorem 2.43 forces \((2.112)\) to hold uniformly on compact subsets of \((0,\infty)\).
We remark that the results from 2.11 are also true in the context of quasi-asymptotic boundedness. Indeed, if one proceeds as in 2.11 but now using the structural theorems of the present section, then one obtains the proofs for the following theorems.

**Theorem 2.44.** Let \( f \in \mathcal{S}'(\mathbb{R}) \). If \( f \) is quasi-asymptotically bounded at 0, with respect to a slowly varying function \( L \), in \( \mathcal{D}'(\mathbb{R}) \), then \( f \) is quasi-asymptotically bounded at 0 of the same degree with respect to \( L \) in the space \( \mathcal{S}'(\mathbb{R}) \).

**Theorem 2.45.** Let \( f \in \mathcal{D}'(\mathbb{R}) \) satisfy \( f(\lambda x) = O(\lambda^\alpha L(\lambda)) \) as \( \lambda \to \infty \) in the space \( \mathcal{D}'(\mathbb{R}) \). If \( \alpha + \beta < -1 \), then \( f \) admits an extension to \( \mathcal{V}_\beta(\mathbb{R}) \) which is equally quasi-asymptotic bounded in the space \( \mathcal{V}'_\beta(\mathbb{R}) \).

**Theorem 2.46.** Let \( f_0 \in \mathcal{D}'(0, \infty) \). Let \( L \) be slowly varying at the origin and \( \alpha \in \mathbb{R} \). Suppose that

\[
\lim_{\varepsilon \to 0^+} f_0(\varepsilon x) = O(\varepsilon^\alpha L(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{in} \quad \mathcal{D}'(0, \infty). \tag{2.113}
\]

Then \( f_0 \) admits extensions to \([0, \infty)\). Let \( f \) be any of such extensions. Then \( f \) has the following asymptotic properties at the origin:

(i) If \( \alpha \not\in -\mathbb{N} \), then there exist constants \( a_0, \ldots, a_n \) such that

\[
f(\varepsilon x) = \sum_{j=0}^{n} a_j \delta^{(j)}(\varepsilon x) + O(\varepsilon^\alpha L(\varepsilon)),
\]

as \( \varepsilon \to 0^+ \) in \( \mathcal{D}'(\mathbb{R}) \).

(ii) If \( \alpha = -k, \quad k \in \mathbb{N} \), then there are constants \( a_0, \ldots, a_n \) and an associate asymptotically homogeneously bounded function of degree 0 with respect to \( L \),

\[
b(ax) = b(x) + O(L(x)), \tag{2.114}
\]

such that

\[
f(\varepsilon x) = \sum_{j=0}^{n} a_j \delta^{(j)}(\varepsilon x) + b(\varepsilon) \delta^{(k-1)}(\varepsilon x) + O(\varepsilon^{-k} L(\varepsilon)), \tag{2.115}
\]

as \( \varepsilon \to 0^+ \) in \( \mathcal{D}'(\mathbb{R}) \).
2.13 Relation between the S-asymptotics and quasi-asymptotics at $\infty$

It is not easy to find conditions for a $T \in \mathcal{F}'$ which imply that $T$ has both the S-asymptotics and quasi-asymptotics. The S-asymptotics is a local property (Theorem 1.4), whereas the quasi-asymptotics has in general a global character (Proposition 2.4, Example 4 in 2.3, and Example 3 in 2.9). Also, we have to find a subspace of $\mathcal{F}'$ in which we can compare these two definitions of the asymptotic behavior and choose the class of functions which will measure the asymptotic behavior following these definitions.

We have seen that the space of tempered distributions is a “natural” one for the quasi-asymptotics (Theorem 2.3, Theorem 2.10 and 2.11.2), while for the S-asymptotics the space $\mathcal{K}'$ has this role (Theorem 1.17). The following examples illustrate the problem of comparison of these two types of asymptotic behavior in $\mathcal{S}'(\mathbb{R})$.

i) The regular distribution $T(t) = H(t) e^{iat}$, $t \in \mathbb{R}$, $a \neq 0$, has the quasi-asymptotics $\frac{i}{a} \delta$ in $\mathcal{S}'(\mathbb{R})$ related to $c(k) = k^{-1}$ (cf. [32]):

\[
\int_{0}^{\infty} e^{ikt} \varphi(t) dt = \frac{1}{ia} \int_{0}^{\infty} \varphi \left( \frac{x}{k} \right) d(e^{iak})
\]

\[
= \frac{-1}{ia} \varphi(0) - \frac{1}{k} \int_{0}^{\infty} e^{iak} \varphi' \left( \frac{x}{k} \right) dx \to \frac{i}{a} \varphi(0), \ k \to \infty.
\]

But the distribution $T$ has no S-asymptotics related to $h^{\alpha}$ with a $U \neq 0$ for any $\alpha \in \mathbb{R}$. We start with

\[
\langle H(t + h) e^{iat}, \varphi(t) \rangle = e^{iah} \int_{-h}^{\infty} e^{iat} \varphi(t) dt
\]

\[
\sim e^{iah} \int_{-\infty}^{\infty} e^{iat} \varphi(t) dt, \ h \in \mathbb{R}_+.
\]

This distribution has the S-asymptotics, but related to the oscillatory function $c(h) = e^{iah}$. 
ii) The regular distribution $T(t) = H(t) \sin t$, $t \in \mathbb{R}$, has the quasi-asymptotics related to $c(k) = k^{-1}$, but it has no S-asymptotics at all:

$$k \langle H(kt) \sin t, \varphi(t) \rangle = \int_0^\infty \sin u \varphi\left(\frac{u}{k}\right) du = \varphi(0) + \frac{1}{k} \int_0^\infty \varphi'\left(\frac{u}{k}\right) \cos u \, du.$$

For the S-asymptotics we have, $h \to \infty$,

$$\langle H(t+h) \sin(t+h), \varphi(t) \rangle = \cos h \int_{-\infty}^\infty \sin t \varphi(t) dt + \sin h \int_{-\infty}^\infty \cos t \varphi(t) dt + o(1).$$

iii) For the regular distribution $T = H(t) \sin \sqrt{t}$, $t \in \mathbb{R}$, we cannot find an $\alpha \in \mathbb{R}$ and a distribution $U_\alpha \neq 0$ such that

$$\lim_{k \to \infty} \int_0^\infty k^\alpha \sin \sqrt{k t} \varphi(t) dt = \langle U_\alpha, \varphi \rangle, \, \varphi \in S(\mathbb{R}).$$

Suppose on the contrary that such $\alpha$ and $U_\alpha$ do exist. We choose $\varphi \in S_+$ such that, $\varphi(t) = e^{-pt}$, $t > 0$, where $\Re p > 0$. Then, we have

$$\lim_{k \to \infty} k^\alpha \int_0^\infty \sin \sqrt{k t} e^{-pt} dt = \langle U_\alpha(t), e^{-pt} \rangle.$$

The value of the last integral is $\sqrt{\pi k/\sqrt{4p^3}} \exp(-k/4p)$ and $\langle U_\alpha(y), e^{-py} \rangle$ is the Laplace transform of $U_\alpha$. Thus the last relation says that the Laplace transform of $U_\alpha$ equals zero for $\Re p > 0$, hence $U_\alpha = 0$.

If we change the basic space, the conclusion can be quite different. Suppose that the basic space is $K_1(\mathbb{R})$.

The function $H(x) \cosh x = \frac{H(x)}{2}(e^x + e^{-x})$, $x \in \mathbb{R}$ defines an element in $K'_1$ and \{exp$(-p x^2)$; $p > 0$\} is in $K_1$.

In order to find the quasi-asymptotics at infinity in $K'_1$ of $H(x) \cosh(x)$, $x \in \mathbb{R}$, we use

$$\int_0^\infty \cosh(kx) e^{-px^2} dx = \frac{1}{2} \int_0^\infty \cosh(k \sqrt{x}) e^{-p x} \frac{dx}{\sqrt{x}} = \frac{\sqrt{\pi}}{\sqrt{4p}} \exp(k^2/4p).$$

This shows that there exists no function $c(k)$ of the form $e^{ak} k^b L(k)$, where $a, b \in \mathbb{R}$, $0 \leq r < 2$, $L(k)$ is a slowly varying function, such that $H(x) \cosh x$ has the quasi-asymptotics related to $c(k)$ with a limit $U \neq 0$. 
For the S-asymptotics, we have
\[
e^{-h} \int_{-h}^{\infty} \cosh(x + h) \varphi(x) dx = \frac{1}{2} e^\varphi(x) dx + \frac{1}{2} e^{-2h} \int_{-h}^{\infty} \varphi(x) dx \to \frac{1}{2} \int_{-\infty}^{\infty} e^\varphi(x) dx, \quad k \to \infty, \quad \varphi \in K_1.
\]

Therefore, \( H(x) \cosh x, x \in \mathbb{R} \), has the S-asymptotics related to \( c(h) = e^h \) with the limit \( U = \frac{1}{2} e^2 \).

The common space in which we can compare the two types of asymptotic behavior is the space of tempered distributions and the common class of functions for the comparison is the class of regularly varying functions \( \{ h^\alpha L(h); \alpha \in \mathbb{R} \} \), where \( L \) is a slowly varying function (cf. 0.3).

The first result of this kind is the following one.

**Proposition 2.23.** (\([32]\)). Let \( g \in S'_+, \alpha > -1 \), and let \( \varphi_0 \in S(\mathbb{R}) \) be such that \( (F \varphi_0)(x) = 1 \) on a neighborhood of zero. If there exists

\[
\lim_{h \to \infty} \langle g(x + h)/h^\alpha, \varphi_0(x) \rangle = C \int \varphi_0(x) dx, \quad C \neq 0,
\]

then \( g(kx) \overset{\mathcal{L}}{\sim} k^\alpha C f_{\alpha+1}(x), \quad k \to \infty, \quad \text{in } S'(\mathbb{R}). \) (For the function \( f_\alpha \) see 0.4).

Later on, this result has been improved (see Theorem 6, Chapter I, §3.3 in \([192]\)) in such a way that, instead of \( h^\alpha \), \( h^\alpha L(h) \) was used, where \( L \) is a slowly varying function. In case \( \alpha \leq -1 \), the situation is quite different. In \([120]\), one can find precise results for \( \alpha = -1 \), but for \( \alpha < -1 \) the obtained results must include the quasi-asymptotic behavior when the limit equals zero, as well.

We quote some results of this kind proved in \([159]\). Denote \( \hat{L}(x) = \int_{a}^{x} \frac{L(t)}{t} dt, x > a. \) If \( \hat{L}(x) \to \infty \) and \( L \) is slowly varying, then \( \hat{L} \) is slowly varying, as well.

**Theorem 2.47.** Suppose that \( T \in S'_+ \) has S-asymptotics related to \( h^\alpha L(h) \). Then \( T \) has also the quasi-asymptotics related to \( h^\beta L'(h) \), where \( \beta \) and \( L' \) are determined as follows: If \( \alpha > -1 \), then \( \alpha = \beta, \ L' = L; \) if
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\(\alpha = -1\) and \(\tilde{L}(x) \to \infty, \ x \to \infty,\) then \(\beta = -1, \ L' = \tilde{L};\) if \(\alpha < -1\) or \(\alpha = -1\) and \(\tilde{L}(x) < \infty,\) then \(\beta = -1, \ L' = \text{const.},\) but the limit can be zero.

Finally, as an illustration of the case \(\alpha < -1\) in Theorem 2.47, we can use the distribution \(T = C_1 \delta^{(r)} + C_2(x^\alpha L(x))_+;\) the quasi-asymptotics depends not only on \(\alpha\) but also on \(r.\)

We can compare the quasi-asymptotics with the S-asymptotics not only for the elements belonging to \(S'+\) but also when they belong to \(D'.\) In this case, we shall use Definition 2.4 instead of Definition 2.2. We know that the general form of the function \(c,\) related to which we can measure the S-asymptotics, is \(c(h) = \exp(\alpha h) L(\exp h), \ \alpha \in \mathbb{R}.\)

**Theorem 2.48.** If \(T \in D'(\mathbb{R})\) has S-asymptotics related to \(c(h) = \exp(\alpha h) \cdot L(\exp h)\) with \(\alpha \neq 0\) and with non-zero limit, then \(T\) cannot have quasi-asymptotics.

At the end of this part dealing with the quasi-asymptotics, we shall cite additionally the following papers: [8], [46], [50], [55], [113], [121], [124], [159], [174], [199] and [201].