Chapter 1

Introduction

During recent years the interest of physicists in nonlocal field theories has been steadily increasing. The main reason for this development is the expectation that the use of these field theories will lead to a much more elegant and effective way of treating problems in particle and high-energy physics as it has been possible up to now with local field theories.

Nonlocal effects may occur in space and time. For example in the time domain the extension from a local to a nonlocal description becomes manifest as a memory effect, which roughly states that the actual behaviour of a given object is not only influenced by the actual state of the system but also by events, which happened in the past.

In a first approach this could be interpreted as an ability of the object to collect or memorize previous events, an idea going back to the physics of Aristotle and perpetuated until the scholastic era. But this concept already irritated Descartes [Descartes(1664)] and seems obsolete since the days of Newton.

In order to allow a better understanding of nonlocal effects, we consider as a simple example the motion of a classical particle in a diluted gas. On the left side of Fig. 1.1 we illustrate the free field case which is characterized by the absence of any boundaries. In that case, with a given collision rate $\psi(t)$, the dynamics of the system is determined by a local theory.

Let us now introduce some walls or boundaries, where the gas molecules bounce off.

This situation is schematically sketched on the right of Fig. 1.1. In that case, at the time $t - \tau$ a fixed rate of gas molecules is scattered, but in contrast to the free field case, at the time $t - \tau/2$ they collide with the boundaries and are reflected. At the time t the dynamics of the system is then characterized by a source term which besides $\psi(t)$ contains an additional nonlocal term proportional to $\psi(t - \tau)$. This is a simple geometric
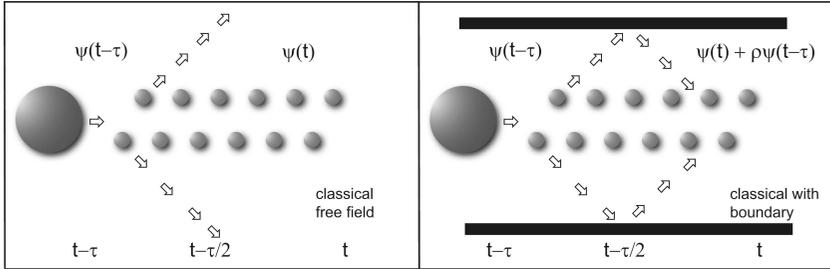
Fig. 1.1.   A possible geometric interpretation of a memory effect. The boundary-free case shown on the left side is described by a local theory (characterized by a source term $\psi(t)$), while the same case including boundaries may be described by a nonlocal theory (characterized by a source term $\psi(t) + \rho\psi(t - \tau)$ with ρ being a proportionality factor).

interpretation of a nonlocal theory. Neither the described particle nor the surrounding medium memorize previous events. There is no intelligence in the system. The boundaries generate a delayed reaction of the medium, which results in a memory effect.

In a more sophisticated scenario, we may assert to every point in space a complex reflection coefficient and are led to Huygens' principle which states that each point of an advancing wavefront is in fact the source of a new set of waves.

In that sense a nonlocal theory may be interpreted as a construct which allows a smooth transition between a local (Newtonian) and a full quantum theory of motion.

A particular subgroup of nonlocal field theories plays an increasingly important role, may be described with operators of fractional nature and is specified within the framework of fractional calculus.

From a historical point of view fractional calculus may be described as an extension of the concept of a derivative operator from integer order n to arbitrary order α, where α is a real or complex value, or even more complicated a complex valued function $\alpha = \alpha(x, t)$:

$$\frac{d^n}{dx^n} \rightarrow \frac{d^\alpha}{dx^\alpha} \tag{1.1}$$

Despite the fact that this concept is being discussed since the days of Leibniz [Leibniz(1695)] and since then has occupied the great mathematicians of their times, no other research area has resisted a direct application for centuries. Abel's treatment of the tautochrone problem [Abel(1823)]

from 1823 stood for a long time as a singular example for an application of fractional calculus.

Not until the works of Mandelbrot [Mandelbrot(1982)] on fractal geometry in the early 1980's was the interest of physicists attracted by this subject, and a first wave of publications in the area of fractional Brownian motion and anomalous diffusion processes was created. But these works lead only to possibly a handful of useful applications and did not produce results of far-reaching consequences [Kilbas(2003)].

The situation changed drastically by progress made in the area of fractional wave equations in recent years. Within this process, new questions in fundamental physics have been raised, that cannot be formulated adequately using traditional methods. Consequently a new research area has emerged, that allows for new insights and intriguing results using new methods and approaches.

The interest in fractional wave equations was amplified in 2000 with a publication by Raspini [Raspini(2000)]. He deduced an $SU(3)$ symmetric wave equation, which turned out to be of fractional nature. In contrast to this formal derivation a standard Yang–Mills theory is merely a recipe for coupling any phenomenologically deduced symmetry. Zavada [Zavada(2002)] has generalized Raspini's result: he demonstrated, that an n-fold factorization of the d'Alembert operator automatically leads to fractional wave equations with an inherent $SU(n)$ symmetry.

In 2002, Laskin [Laskin(2002)] on the basis of the Riesz definition [Riesz(1949)] of the fractional derivative presented a Schrödinger equation with fractional derivatives and gave a proof of Hermiticity and parity conservation of this equation.

In 2005, the Casimir operators and multiplets of the fractional extension of the standard rotation group $SO(n)$ were calculated algebraically [Herrmann(2005)]. A mass formula was derived, which successfully described the ground state masses of the charmonium spectrum. This may be interpreted as a first approach to investigate a fractional generalization of a standard Lie algebra, a first attempt to establish a fractional group theory and the first non-trivial application of fractional calculus in multidimensional space.

In 2006, Goldfain [Goldfain(2006)] demonstrated, that the low level fractional dynamics [Tarasov(2006)] on a flat Minkowski metric most probably describes similar phenomena as a field theory in curved Riemann space time. In addition, he proposed a successful mechanism to quantize fractional free fields. Lim [Lim(2006a)] proposed a quantization description for

free fractional fields of Klein–Gordon-type and investigated the temperature dependence of those fields.

In 2007, we [Herrmann(2007)] applied the concept of local gauge invariance to fractional free fields and derived the exact interaction form in the first order of the coupling constant. The fractional analogue of the normal Zeeman-effect was calculated and as a first application a mass formula was presented, which gives the masses of the baryon spectrum an accuracy better than 1%. It has been demonstrated, that the concept of local gauge invariance determines the exact form of interaction, which in the lowest order coincides with the derived group chain for the fractional rotation group.

Since then the investigation of the fractional rotation group alone within the framework of fractional group theory has lead to a vast amount of interesting results, e.g. a theoretical foundation of magic numbers in atomic nuclei and metallic clusters.

It is the explicit intention of this book to give a concise introduction to this new vividly growing branch of fractional calculus.

This book is an invitation to the interested student and to the professional researcher as well. It presents a thorough introduction to the basics of fractional calculus and routes the reader up to the current state-of-the-art physical interpretation.

What makes this textbook unique is its application oriented approach. If you are looking merely for a mathematical foundation of fractional calculus we refer to the literature (e.g. [Oldham(1974), Samko(1993b), Miller(1993), Kiryakova(1994), Rubin(1996), Gorenflo(1997), Podlubny(1999), Hilfer(2000, 2008), Mainardi(2010)]). This book is explicitly devoted to the practical consequences of using fractional calculus.

The book may be divided into two parts. Chapters 2 to 8 give a step by step introduction to the techniques and methods derived in fractional calculus and their application to classical problems.

Chapters 9 to 19 are devoted to a concise introduction of fractional calculus in the area of quantum mechanics of multi-particle systems. The application of group theoretical methods will lead to new and unexpected results. The reader is directly led to the actual state of research.

All derived results are directly compared to experimental findings. As a consequence, the reader is guided on a solid basis and is encouraged to apply the fractional calculus approach in his research area, too.

It will be demonstrated, that the viewpoint of fractional calculus leads to new insights and surprising interrelations of classical fields of research that remain unconnected until now.