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On the Mathematics Lesson

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1 Introduction

The basic form of mathematics instruction in Russia is the classroom lesson. Of course, other forms exist, such as individual instruction, which is used with poorly performing students after classes or with seriously ill children at home. Naturally, teaching also takes place outside of class — through extracurricular activities, homework, and so on — so it cannot be equated with that which goes on in class. Nonetheless, it would be no mistake to repeat that the basic form of mathematics instruction is the classroom lesson. Not for nothing were those who in Soviet times were most concerned with teaching students, and not with what was conceived of as communist character-building work, contemptuously labeled “lesson providers.”

Every day in the upper grades, six or seven required classes are followed by optional activities outside of the standard schedule: elective
classes, and special courses in various subjects, including those not part of the standard school program — in other words, effectively more classes (but, in contrast to those that are part of the standard schedule, these are not mandatory for everyone). Over the course of his or her schooling, a student attends about 2000 mathematics classes, while a mathematics teacher teaches several tens of thousands of classes throughout his or her career (Ryzhik, 2003). Consequently, much has been written and discussed about planning and conducting classes, in all subjects in general and in mathematics in particular. Dozens of manuals on conducting classes have been developed and published, presenting problems for solving in class, quizzes for testing students in class, and simply lesson plans. Even today, despite the availability of numerous publications and possibilities for copying necessary materials, lectures in which an experienced teacher presents and discusses various approaches to conducting lessons remain popular.

This chapter is devoted to the lesson and how it is constructed and conducted in Russian mathematics classrooms. Of course, it is impossible to talk about any system for conducting classes in mathematics that is common to all Russian (Soviet) teachers: the country is large, and although the same requirements apply everywhere and control has sometimes been very rigid, the diversity of the lessons has been and remains great. Sometimes, lessons conducted in accordance with official requirements have been very successful; on other occasions, although they apparently followed the rules, some lessons have clearly turned out badly. Additionally, analyses of lessons conducted by mathematics supervisors even in Stalin’s time include numerous remarks suggesting that classes were not conducted according to the requirements. Nonetheless, the very existence of common requirements leads us to reflect on some common characteristics of Russian mathematics lessons. Many of these characteristics emerged during the 1930s–1950s — the formative years of Soviet schools — after almost all post-Revolution explorations were rejected. We will therefore discuss the methodological works of this period, gradually progressing into modern times. But first, to provide some background, we will say a few words about the conditions under which classes are conducted today.
2 Who Participates in the Class and Where Classes Are Conducted: Background

To provide a better understanding of the specific character of Russian classes, we must describe certain important features of the way in which the teaching process has been organized in Russia, both traditionally and at the present time.

2.1 Teachers and Students

Perhaps the most important difference between the teaching of mathematics in Russia and, say, in the United States is the fact that usually a teacher works with the same class for a considerable length of time — the composition of the class virtually does not change, and the class continues to have the same teacher. Instruction is broken down not into different courses that the students can take, but simply into different years of schooling — in fifth grade everyone studies specific topics, and in sixth grade everyone moves on to other topics. A teacher can be assigned to a fifth-grade classroom and, in principle, remain with the students until their graduation (note that in Russia there is no distinction between middle and high school in the sense that students of all ages study in the same building, have the same principal, and so on).

One of the authors of this chapter, for example, had the same mathematics teacher during all of his years in school, from fifth grade until his final year (which, at the time, was tenth grade). The composition of the class did not change much either. Of course, there were “new kids” who would come from other schools, usually because their families had moved. And, of course, some students left, usually again because their families moved or (very few) because they transferred to less demanding schools (such transfers often took place after students completed what today is called the basic school, which at that time ended with eighth grade — students would then transfer to vocational schools, for example). However, the overwhelming majority of the class remained together from first grade until tenth grade.

Today, while the mobility of the population is somewhat greater, it may be confidently asserted that in an ordinary school the students in
a class usually know one another for at least several years. Moreover, although it is now less common for the same teacher to take a class from beginning to end, a teacher will still usually remain with the same class for at least a few years. Naturally, it would not be difficult to point out certain shortcomings of this system, in which the image of the teacher can almost be equated with the image of mathematics — and this is hardly a good thing, particularly if the teacher is not a good teacher. At the same time, certain advantages of this system remain evident: teachers know their classes well, and the classes have time to become accustomed to their teachers’ demands; long-term planning in the full sense of the word is feasible, as teachers themselves prepare students for what they will teach in the future. Moreover, such a system in some measure makes the results of a teacher’s work more obvious: it would be wrong always to blame the teacher for a poorly prepared class, but at the same time it would definitely be impossible to blame other teachers because, in short, there were no other teachers.

The required number of students in a class has decreased as the Russian school system has developed. If a class in the 1960s had 35–40 students, now, as a rule, it consists of 25–30 students (here, we are not considering the so-called schools with low numbers of students: a distinctive phenomenon in Russia, where given the existence of tiny villages scattered at great distances from one another it was necessary — and here and there remains necessary — to maintain small schools whose classes could have as few as two or three students).

Elementary school students have the same teacher for all subjects (with the exception of special subjects such as music and art). Teachers of elementary school classes are prepared by special departments at pedagogical institutes and universities as well as special teachers’ colleges. The main problem with classes in the teaching of mathematics in elementary school is that not all teachers will have devoted sufficient time to studying mathematics in their past school or college experience, not all teachers regard this subject with interest, and not all teachers have a feeling for its unique character and methodology. The teaching of mathematics can therefore turn into rote learning of techniques, rules, or models for writing down solutions, thereby fostering negative reactions in children between the ages of 7 and 10 and suggesting to
them that the main instrument for studying mathematics is memory, not logical reasoning and mental agility. In recent times, the problems associated with mathematics instruction in elementary schools have finally started to receive more active attention from better-prepared experts in mathematics education.

Beginning with fifth grade, mathematics classes are taught by specialist subject teachers who have graduated, as a rule, from the mathematics department either of a pedagogical institute or a university. In today’s schools, one also finds former engineers who have lost their jobs for economic reasons and have become re-educated, in some comparatively short program of study, as teachers.

The hours allocated in each class for mathematics consist of the so-called federal — i.e. stipulated by the Ministry of Education — component and other components determined by the region and, to some extent, by the school itself. The number of mathematics classes per week can thus vary both for different years of study and for different schools. Nevertheless, usually in the so-called ordinary class (i.e. a class without advanced study of mathematics and without advanced study in the humanities), 5–6 hours per week are devoted to mathematics. One lesson usually lasts 45 minutes, although in certain periods and in certain schools there have been and continue to be experiments in this respect as well — a 40-minute lesson, a 50-minute lesson, and so on. From seventh grade on, mathematics is split into two subjects: geometry (grades 7–11) and algebra (grades 7–9) or algebra and elementary calculus (grades 10–11).

Students’ mathematical preparedness can vary greatly. A diagnostic study conducted by one of the authors of this chapter in two districts of St. Petersburg in 1993 (Karp, 1994) revealed that approximately 40% of tenth graders were unable to complete assignments at the ninth-grade level, while 30% got top grades on such assignments, and approximately 3.5% displayed outstanding results in solving difficult additional problems. (We cite this old study because we believe, for a number of reasons, that its results, at least at the time of the study, accurately reflected the existing state of affairs. At the same time, it must be noted that a very famous school with an advanced course of mathematics was located in one of the districts studied, which naturally
would have somewhat improved the average results by comparison with the average level in the whole city.)

The same study revealed a noticeable spread between different schools and classes: in some classes (including classes even outside the aforementioned school), virtually all students received top scores on their assignments, but in other classes none of the students were able to do the work. It is likely that such differences became more profound in subsequent years. At the same time, these differences were not related, as sometimes happens in the United States, for example, to whether the schools were located in the inner city or in the suburbs. Naturally, schools with an advanced course of study in mathematics admit students with a somewhat higher level of preparation. Moreover, in schools with an advanced course of any kind (such as schools with an advanced course in the English language), the average level of mathematics is usually somewhat higher than in ordinary schools. But, not infrequently, ordinary schools with strong teaching and administrative staffs — i.e. schools already having comparatively well-prepared teachers — would go on to become specialized schools with advanced courses of study in various subjects.

In any case, students’ levels in, say, a seventh-grade classroom can vary greatly; the same is true even of a tenth-grade classroom (by tenth grade, the most capable students might have already transferred to schools with an advanced course in mathematics and the least interested students would have transferred, for example, to vocational schools). In a class, the teacher sometimes must simultaneously challenge the most gifted students without focusing on them exclusively; select manageable assignments for the weakest students and do as much as possible with them; and work intensively with so-called “average” students, considering their individual differences and selecting the most effective techniques for teaching them.

### 2.2 The Mathematics Classroom and Its Layout

The mathematics classroom, a special classroom in which mathematics classes are conducted, has usually seemed barren and empty to foreign visitors. They see no cabinets filled with manipulatives, no row of
computers next to the wall or in the back of the room, no tables nearby piled high with materials of some kind or other. There is no Smartboard and most likely not even an overhead projector.

The large room has three rows of double desks, and each double desk has two chairs before it. The desks are not necessarily bolted down, but even so, no one moves them very much — the students work at their own desks. The front wall is fully mounted with blackboards. Usually, the mathematics teacher asks the school to set up the blackboards in two layers at least on a part of the wall; this would allow the teacher to write on one board and then shift it over to continue writing or to open up a new space with text already prepared for a test or with answers to problems given earlier. Various drawing instruments usually hang beside the blackboards. There may also be blackboards on the side and rear walls of the classroom. Discussing completed assignments on a rear-wall board is not very convenient, because the students must turn around; however, such a blackboard can be reserved for working with a smaller group of students while the rest of the class works on another assignment. The teacher’s desk is positioned either in front of the middle row of desks facing the students, or on the side of the classroom against the wall.

Mathematical tables hang on the classroom walls. Usually, these are tables of prime numbers from 2 to 997, tables of squares of natural numbers from 11 to 99, and tables of trigonometric formulas (grades 9–11). The classroom has mounting racks that can be used to display other tables or drawings as needed (such as drawings of sections of polyhedra when studying corresponding topics). Mathematical tables are published by various pedagogical presses, but they may also be prepared by the teachers themselves along with their students. (Recently, paper posters have started getting replaced with computer images which can be displayed on large screens, but for the time being these remain rare.)

On the same racks may be displayed the texts of the students’ best reports, sets of Olympiad-style problems for various grades, along with lists of students who first submitted solutions to these problems or with their actual solutions, problems from entrance exams to colleges that students are interested in attending, or problems from the Uniform
State Exam (USE), and so on. A virtually obligatory component of mathematics classroom decoration consists of portraits of great mathematicians. Usually, these include portraits of such scientists as François Viète, Carl Friedrich Gauss, David Hilbert, René Descartes, Sofia Vasilyevna Kovalevskaya, Andrey Nikolaeivich Kolmogorov, Gotthfried Wilhelm Leibniz, Nikolay Ivanovich Lobachevsky, Mikhail Vasilievich Ostrogradsky, Henri Poincaré, Leonhard Euler, Pafnuty Lvovich Chebyshev, and Pierre Fermat. In class, the teacher might talk about one or another scientist, and draw the students’ attention to his or her portrait.

Usually, the classroom features bookcases with special shelves dedicated to displaying models of geometric objects and their configurations. Students might have made these models out of paper. For difficult model-construction projects lasting many hours, students may refer to M. Wenninger’s book *Polyhedron Models* (1974); for preparing simpler models, they can rely on the albums of L. I. Zvavich and M. V. Chinkina (2005), *Polyhedra: Unfoldings and Problems*. Students having such albums may be given individual or group home assignments to construct a paper model of, say, a polyhedron with certain characteristics and then to describe the properties and features of this polyhedron while demonstrating their model in class. Such student-constructed models may include, for example, the following: a tetrahedron, all of whose faces are congruent scalene triangles; a quadrilateral pyramid, two adjacent faces of which are perpendicular to its base; a quadrilateral pyramid, two nonadjacent faces of which are perpendicular to its base (note that constructing such a model may be difficult but also very interesting for the students); and so on. Any one of these models can be used for more than one lesson of solving problems and investigating mathematical properties. Factory-made models of wood, plastic, rubber, and other materials may also be on display in the classroom. During particular lessons, these models may be demonstrated and studied. Using models for demonstrations differs from using pictures for the same purpose, owing to the higher degree of visual clarity that the former provide, since models can be constructed only if objects really exist, while pictures can even represent objects that do not exist in reality. In contrast to pictures,
models allow students not only to see but also to “feel” geometric objects.

Sets of cards for individual questions during class, prepared by teachers over many years, are stored on special shelves in the cabinets. Also kept in the cabinets are notebooks for quizzes and tests. An extremely important part of the classroom may be its library located in the classroom bookcases. In this respect, of course, much depends on the tastes and interests of the teacher (especially since the school usually provides little or no support for creating a library). Meanwhile, the presence of books in the classroom is helpful not only because they may be used during classes or given to students for independent reading at home or for preparing reports, but also because students learn to read and love books about mathematics when teachers talk about, demonstrate, and discuss books.

The library may contain binders of articles from the magazine *Kvant*, books from the popular series “The Little *Kvant* Library,” pamphlets from the series “Popular Lectures in Mathematics,” and so on. On the other hand, such libraries frequently contain collections of tests and quizzes, educational materials for various grades in algebra and geometry, as well as sets (approximately 15–20 copies) of textbooks and problem books in school mathematics. With multiple copies, students will have the books they need to work at their own desks, while teachers can conduct classes (or parts of classes) including students’ work on theoretical materials from one or another textbook or manual or their work on solving problems from one or another problem book. In the past, when teachers had no way of copying the necessary pages, having multiple copies of books was especially important — even now, though it is often more convenient to work with an entire problem book than with a set of copied pages.

Independent classroom work with theoretical materials from the textbook is also extremely important. Helping students develop the skill of working with a book is one of the teacher’s goals. Students rarely develop this skill on their own; for this reason, it is desirable for teachers to create conditions in which students will need to call upon this skill, and teachers will be able to demonstrate how to work with a book. For example, a teaching manual containing solutions to various
problems, such as V. V. Tkachuk’s book *Mathematics for the Prospective College Student* (2006), may be distributed to the students before class, and they may be asked to use it to examine the solution to a problem of medium difficulty involving parametric variables. One can go further and organize a lesson around a discussion on different methods for constructing proofs. Various geometry textbooks may be chosen as materials for this purpose, with students being asked to compare the different techniques employed in them to prove the Pythagorean theorem (grades 8 and 9); to prove that certain conditions are sufficient for a straight line to be perpendicular to a plane (grade 10); or to derive the formula for the volumes of solids of revolution (grade 11). Such lessons are difficult to prepare, but they are extremely informative and useful. However, they are not feasible in all classes, but only in classes with sufficiently interested students.

In sum, we would say that the mathematics classroom has usually had, and indeed continues to have, a spartan appearance not only because Russian schools are poor (although, of course, the lack of funds is of importance: some schools that for one or another reason have more money can have Smartboards, magic markers instead of chalk for writing on the board, and many computers, although this does not necessarily suggest that the computers are being used in a meaningful way). The view is that students should not be distracted by anything extraneous during class. Class time is not a time for leisurely looking around, but for intensive and concentrated work.

### 3 Certain Issues in Class Instruction Methodology

#### 3.1 On the History of the Development of Class Instruction Methodology in Russia

The collection of articles entitled *Methodology of the Lesson*, edited by R. K. Shneider (1935), opens with an article by Skatkin and Shneider (1935) which contrasts the contemporary Soviet lesson with both the type of lesson preceding the Revolution and the one immediately following it and reflecting “left-leaning perversions in methodology.” As an example of pre-Revolution schooling, the authors present a lesson about a dog (probably for elementary school students), supposedly
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taken from teaching guidelines set in 1862. During this lesson, the teacher was supposed to systematically give answers to the following questions: “What is a dog?,” “How big is the dog?,” “What is it covered with?,” “What kind of fur does the dog have?,” and so on. After this, the students themselves were to use the same questions to tell about the dog. The authors conclude: “No mental work is required to master such content; there is nothing to think about here, since there are no relations, connections, causes, explanation” (p. 4). As for the “left-leaning” lessons, the authors describe a class officially devoted to the poem “The Starling” by the great Russian fable writer Ivan Krylov, during which the teacher launched into a discussion with the students about whether they had ever seen a starling, why starlings are useful, why people build birdhouses, and so on. In this way, the meaning of the fable for the study of Russian language and literature was, in the authors’ opinion, lost.

The “left-leaning” system was criticized for not pursuing the goal of giving the children a “precisely defined range of systematic knowledge.” As an example of arguments directed against knowledge, the authors cite the German pedagogue Wilhelm Lamszus (1881–1965):

How much of what you and I memorized by rote in mathematics can we really apply in life? All of us, I recall, tirelessly, to the point of fainting, studied fractions, added and subtracted, multiplied and divided proper and improper fractions, all of us diligently converted ordinary fractions into decimals and back again. And now? What has remained of all this? Indeed, what mathematics does a young woman need to know when she becomes a housewife in order to run a household successfully? (p. 6)

Concluding (and largely with reason, it would seem) that such an orientation against knowledge in reality conceals the view that certain portions of the population do not need knowledge, Skatkin and Shneider proceed to formulate a set of requirements for lessons. Among them is the requirement that both the knowledge conveyed and the lessons devoted to it be systematic: “A lesson must be organically connected with the lesson before it and prepare the way for the lesson

1This and subsequent translations from Russian are by Alexander Karp.
after it” (p. 8). The requirement of precision is also emphasized, in relation to both the objectives and the conclusions of each lesson. The unity of content, of methodological techniques, and of the structure of the lesson as a whole is put forward as another requirement. In this regard, the authors propose replacing rote learning of the content with conscious and critical acquisition and assimilation.

If Skatkin and Shneider’s article was devoted to a theoretical conceptualization of the problem, then L. V. Fedorovich’s (1935) article in the same collection gives recommendations (or perhaps even issues orders) about implementing the formulated requirements in practice. Fedorovich writes as follows:

All of the work must be structured in a way that allows the teacher to pass from the practical problem, the concrete example, to the general law, and after studying the general law with the class, once again to illustrate its application in solving practical problems. (p. 119)

The description of how a lesson must be structured and taught is rigid and precise. For example, the lesson must begin in the following way:

Everything is prepared for the beginning of the class. The students enter in an organized fashion. All of them know their places (seating is fixed), so there is no needless conversation, above all, no arguments about seats. The students must be taught to prepare their notebooks, books, and other personal materials in 1–2 minutes.... The moment when the class is ready is signaled by the teacher, and the students begin to work. (p. 120)

The next recommended step is the checking of homework assignments (the teacher conducts a general discussion and also examines students’ notebooks). The teacher must also demonstrate how to complete, and how not to complete, the assignments. All of this should consume 8–12 minutes.

In studying new material, the author recommends:

- Clearly formulating the aim of the lesson for the students;
- Connecting the new lesson with the preceding lesson;
- Identifying the central idea in the new material, paying particular attention to it;
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- Viewing the lesson as a link in a unified system and consequently adhering to the common analytic approach;
- Including elements of older material in the presentation of new material;
- Reinforcing the new material;
- Following the textbook in presenting the material.

As for the techniques to be used in presenting the material, the author states that “the techniques must be varied in accordance with the nature of the material itself, the textbooks, and the class’s level of preparedness” (p. 124). The author further recommends “mobilizing visual, auditory, and motor perception,” using various ways to work with students (verbal communication by the teacher, demonstration, laboratory work, exercises, mental arithmetic, independent work, and so on) and, specifically, using tables and visual aids. Special recommendations are provided on how to avoid mechanical memorization, work on proving theorems, and teach students to construct diagrams (examples show how these should and should not be constructed).

At the end of the class, the teacher summarizes the material, draws conclusions (such as by asking: “What is the theorem that we have examined about?”), and assigns homework. Further, the article indicates that the students are to write down this assignment in their notebooks, tidy up their desks, and leave the classroom in an organized fashion.

To carry out these recommendations, teachers needed to be good at selecting substantive assignments for their students, which was not always the case in practice. At least, the importance of posing substantive questions and recognizing that not everyone was capable of doing so subsequently became a much-discussed topic. For example, an article entitled “Current Survey” (Zaretsky, 1938) published in the newspaper *Uchitel’skaya gazeta* (Teachers’ Newspaper) contains numerous recommendations about how to pose and how not to pose questions in class:

Suppose the students have studied the properties of the sides of a triangle. Why not ask them the following: one side of a triangle is 5 cm long, another is 7 cm long; how long might the third side be?
The same article recommends posing questions that are formulated differently from how they are in the textbook: “Thus, in geometry, a student may be asked to give an explanation based on a new diagram.” These and other techniques aimed to prevent purely formal memorization of the material. However, judging by the fact that the need to fight against empty formalism in learning remained a subject of discussion for several decades, it was not always possible to implement the recommendations easily and successfully in real life.

On the other hand, the rigidity of the methodological recommendations, even if they were reasonable, could itself cause harm, depriving teachers of flexibility (it should be borne in mind that the implementation of methodological recommendations was often monitored by school administrators who did not always understand the subject in question). As a result, during the 1930s, a rigid schema evolved for the sequence of activities during a lesson: (a) homework review; (b) presentation of new content; (c) content reinforcement; (d) closure and assignment of homework for the next lesson. Going into slightly more detail, we may say that the vast majority of lessons, which always lasted 45 minutes, were constructed in the following manner:

**Organizational stage** (2–3 minutes). The students rise as the teacher enters the classroom, greeting him or her silently. The teacher says: “Hello, sit down. Open your notebooks. Write down the date and ‘class work.’” The teacher opens a special class journal, which lists all classes and all grades given in all subjects, and indicates on his or her own page of the journal which students are absent. On the same page, the teacher writes down the topic of the day’s lesson and announces this topic to the students.

**Questioning the students, checking homework, review** (10–15 minutes). Three to five students are called up to the blackboard, usually one after another but sometimes simultaneously, and asked to tell about the material of the previous lesson, show the solutions to various homework problems, talk about material assigned for review, and solve exercises and problems pertaining to material covered in the previous lesson or based on review materials.

**Explanation of new material** (10–15 minutes). The teacher steps up to the blackboard and presents the new topic, sometimes making use of materials from a textbook or problem book in the presentation.
Until the early 1980s, the same set of mathematics textbooks was used throughout Russia. Sometimes, instead of explaining new material to the students, the teacher asks them to work with a text, and the students read and write an outline of the textbook.

Reinforcement of new material, problem solving (10–15 minutes). The students open their problem books and solve the problems assigned by the teacher. Usually, three or four students are called up to the blackboard, one after another.

Summing up the lesson, homework assignment (2–3 minutes). The teacher sums up the lesson, reviews the main points of the new material covered, announces students’ grades, reveals the topic of the next lesson and the review topic, and assigns homework — which as a rule corresponds to a section from the textbook that covers the new material, sections from the textbook that cover topics for review, and problems from the problem book that correspond to the new material and review topics.

By the 1950s, this schema was already, even officially, regarded as excessively rigid. A lead article in the magazine *Narodnoye obrazovanie* (*People’s Education*), praising a teacher for his success in developing in his students a sense of mathematical literacy, logical reasoning skills, and “the ability not simply to solve problems, but consciously to construct arguments,” explained the secret behind his accomplishments:

Boldly abandoning the mandatory four-stage lesson structure whenever necessary, the pedagogue constantly searched for means of activating the learning process. He was “not afraid” to give the students some time for independent work, when this was needed, sometimes even the lesson as a whole, both while explaining new material and while reinforcing their knowledge (Obuchenie, 1959, p. 2).

It is noteworthy, however, that in order to do so, the teacher had to act boldly.

And yet, although the commanding tone of the recommendations cited at the beginning of this section cannot help but give rise to objections, it must be underscored that the problem posed was the problem of constructing an intensive and substantive lesson — a lesson in which the possibility of obtaining a deep education would
be offered to all students. This last fact seems especially important. It would be incorrect, of course, to think that Soviet schools successfully taught 100% of their students to prove theorems or even to simplify complicated algebraic formulas — the number of failing students in a class might have been as high as 20%, and far from all students went on to complete the upper grades. Nonetheless, the issue of familiarizing practically all students with challenging mathematics which contained both arguments and proofs was at least considered.

Again, this issue was not always resolved successfully in practice. When the following bit of doggerel appeared in a student newspaper:

There’s no order in the classrooms,
We can do whatever we please.
We don’t listen to the teacher
And our heads are in the clouds.

it was immediately made clear that such publications were politically harmful [GK VKP(b), 1953, p. 7]. It may, however, be supposed that discipline in the classroom was indeed not always ideal. Inspectors who visited classes [for example, GK VKP(b), 1947] noted the teachers’ lack of preparation and their failure to think through various ways of solving the same problems; the students’ inarticulateness and the teachers’ inattentiveness to it; and the insufficient difficulty of the problems posed in class and poor time allocation during the lesson.

The reports of the Leningrad City School Board pointed out the following characteristic shortcomings of mathematics classes:

- Lessons are planned incorrectly (time allocation).
- Unacceptably little time is allocated for the presentation of new material.
- The ongoing review of student knowledge is organized in an unsatisfactory fashion — students are rarely and superficially questioned, while homework is checked inattentively and analyzed superficially.
- Systematic review is lacking.
- Work on the theoretical part of the course is weak — conscious assimilation of theory is replaced by mechanical memorization
without adequate comprehension. Teachers are inattentive to students’ speech.

- Insufficient use is made of visual aids and practical applications.
- Students’ individual peculiarities and gaps in knowledge are poorly studied (LenGorONO, 1952, p. 99).

In other words, practically all of the recommendations cited above met with violations and obstacles. Nonetheless, the unflagging attention to these aspects of the lesson in itself deserves attention.

### 3.2 Types of Lessons and Lesson Planning

The recognition that constructing all lessons in accordance with the same schema is neither always possible nor effective led to the identification of different types of lessons and to the formation of something like a classification of these different types of lessons. Considerable attention has been devoted to this topic in general Russian pedagogy and, more narrowly, in the methodology of mathematics education. Manvelov (2005) finds it useful to identify 19 types of mathematics lessons. Among them — along with the so-called combined lesson, the structure of which is usually quite similar to the four-stage schema described above — are the following:

- The lesson devoted to familiarizing students with new material;
- The lesson aimed at reinforcing what has already been learned;
- The lesson devoted to applying knowledge and skills;
- The lesson devoted to generalizing knowledge and making it more systematic;
- The lesson devoted to testing and correcting knowledge;
- The lecture lesson;
- The practice lesson;
- The discussion lesson;
- The integrated lesson; etc.

As we can see, several different classifying principles are used here simultaneously. The lecture lesson, for example, may also be a lesson devoted to familiarizing students with new material. We will
not, however, delve into theoretical difficulties here; they may be unavoidable when one attempts to encompass in a general description all of the possibilities that are encountered in practice. Instead, we will offer examples of the structures of different types of lessons.

A lesson devoted to becoming familiar with new material that deals with “the multiplication of positive and negative numbers,” examined by Manvelov (2005, p. 98), has the following structure:

1. Stating the goal of the lesson (2 minutes);
2. Preparations for the study of new material (3 minutes);
3. Becoming acquainted with new material (25 minutes);
4. Initial conceptualization and application of what has been covered (10 minutes);
5. Assigning homework (2 minutes);
6. Summing up the lesson (3 minutes);

For comparison, the practice lesson has the following structure:

1. Stating the topic and the goal of the workshop (2 minutes);
2. Checking homework assignments (3 minutes);
3. Actualizing the students’ base knowledge and skills (5 minutes);
4. Giving instructions about completing the workshop’s assignments (3 minutes);
5. Completing assignments in groups (25 minutes);
6. Checking and discussing the obtained results (5 minutes);

We will not describe the assignments that teachers are supposed to give at each lesson; thus, our description of the lessons will be limited, but the difference between the lessons is nonetheless obvious. Even greater is the difference between them and such innovative types of lessons as the discussion lesson or the simulation exercise lesson, which we have not yet mentioned and which is constructed precisely as a simulation exercise (as far as we can tell, this type of mathematics lesson is, at least at present, still not very widespread). In contrast to the two lessons described above, in which some similarities to the traditional four-stage lesson can still be detected, the innovative types of lessons altogether differ from any traditional approach.
Naturally, the objectives of a lesson dictate which type of lesson will be taught, and the objectives of the lesson are in turn dictated by the objectives of the teaching topic being covered and by the objectives of the course as a whole. In practice, this means that the teacher prepares a so-called *topic plan* for each course. More precisely, teachers very often do not so much prepare topic plans on their own as adapt the plans proposed by the Ministry of Education. The Ministry proposes a way to divide class hours among the topics of the course, while using one or another Ministry-recommended textbook. Sometimes, teachers use this plan directly; sometimes, they alter the distribution of hours (for example, adding hours to the study of a topic if more hours have been allocated for mathematics at their school than the Ministry had stipulated). In theory, a teacher today has the right to make more serious alterations; but, in practice, the possibilities of rearranging the topics covered in the course are limited — the students already have the textbooks ordered by their school in their hands. Rearranging topics will most likely undermine the logic of the presentation, so the only teachers who dare to make such alterations either are highly qualified and know how to circumvent potential difficulties or are unaware that difficulties may arise. (In fact, district or city mathematics supervisors have the right not to approve plans, but at the present time this right is not always exercised.)

Subsequently, the teacher proceeds to planning individual lessons. Note that it has been a relatively long time since the preparation of a written lesson plan as a formal document was officially required; the plan is now seen as a document for the teacher’s personal use in his or her work. At one time, however, a teacher lacking such a document might not have been permitted to teach a class, with all the consequences that such a measure entailed. School administrators frequently demanded that lesson plans be submitted to them and they either officially approved or did not approve them.

Generally speaking, if, say, four hours are allocated for the study of a concept, then the first of these hours will most likely contain more new material than subsequent hours and, therefore, may be considered a lesson devoted to becoming familiar with new material. During the second and third classes, there will probably be more problem-solving,
and so those lessons may be considered practice lessons. And the fourth lesson may likely be considered a lesson devoted to testing and correcting knowledge.

Again, however, reality can destroy this theoretical orderliness: new material can (not to say must) be studied in the process of solving problems, and therefore it is not always easy to separate becoming familiar with new material from doing a practice on it. The demand that content, methodological techniques, and the structure of the lesson as a whole be unified, as Skatkin and Shneider (1935) insisted, can be fully satisfied only when there is a sufficiently deep understanding of both what the mathematical content of the lesson might look like and how the lesson might be structured (Karp, 2004). In particular, it is necessary to gain a deeper understanding of the role played in class by problem solving and by completing various tasks in general. It is to this question that we now turn.

4 Problem Solving in Mathematics Classes

The methodological recommendations of the 1930s and the subsequent years are full of instructions that the teacher’s role must be enlarged. Indeed, teachers were seen as captains of ships, so to speak, responsible for all that occurs in the classroom while at the same time enjoying enormous power there (to be sure, they were endowed with this power as representatives of an even higher power, to which they in turn had to submit, in principle, completely). Teachers were regarded as organizers or, better, designers of lessons, although it would be incorrect automatically to characterize Russian teaching as “teacher-centered” — to use a contemporary expression — especially since this expression usually requires additional clarification. It would be a mistake to equate the dominant role of the teacher as a designer of the lesson, for example, with the lecture style of presentation, or even with a teacher’s monopoly of speaking in class. Ideally, the teacher would select and design problems and activities that would enable the students to become aware of new concepts on their own; to proceed gradually and independently from simple to difficult exercises and to further theoretical conceptualization; to think on their own about applying
what they have learned; to discover their own mistakes; and so on. This did not rule out that the teacher himself or herself usually posed the questions, summarized the material, or provided the theoretical foundation for various problems.

The mathematics class in the public consciousness was a place where students were taught to think, and this was intended to be achieved through problem solving. In classes devoted to subjects in the natural sciences (physics, chemistry, etc.), the experiment occupies a very important position, and it is precisely in the course of the experiment and the discussion of its organization and results that a student’s interests in the subject are formed and developed. In mathematics, then, the equivalent of the experiment is in a sense problem solving. An entire course in mathematics can in fact be constructed — and often is constructed — around the solving of various problems of different degrees of importance and difficulty. Clearly, any theorem may and should be regarded as a problem, and its proof as the solution to that problem. Likewise, the theorem’s various consequences should be seen as applications of that problem.

As an example, let us examine one of the most difficult theorems in the course in plane geometry designed by L. S. Atanasyan et al. (see, for instance, Atanasyan et al., 2004): the theorem concerning the relations between the areas of triangles with congruent angles. This theorem states that, if an angle in triangle $ABC$ is congruent to an angle in triangle $A_1B_1C_1$, then the areas of the two triangles stand in the same relation to each other as the products of the lengths of the sides adjacent to these angles. In other words, if, for example, angle $A$ is congruent to angle $A_1$, then $\frac{A_{ABC}}{A_{A_1B_1C_1}} = \frac{AB \cdot AC}{A_1B_1 \cdot A_1C_1}$ ($A$ is the area). This theorem is very important, since it is then used to prove that certain conditions are sufficient for triangles to be similar, which in turn serves as the basis for introducing trigonometric relations and so forth. In and of itself, too, this theorem makes it immediately possible to solve a number of substantive problems (which will be discussed below). At the same time, its proof is not easy for schoolchildren, and the actual fact that is proven looks somewhat artificial (why should areas be connected with the relations between sides?). The teacher can structure a lesson so that the students themselves ultimately end up
proving the required proposition by solving problems that seem natural to them. For example, the teacher may offer the following sequence of problems:

1. Point $M$ lies on the side $AB$ of triangle $ABC$. $\frac{BM}{AB} = \frac{1}{3}$. It is known that the area of triangle $ABC$ is equal to $12 \text{ cm}^2$. What is the area of triangle $BMC$?

2. Under the conditions of the previous problem, let there be given an additional point $K$ on side $BC$, such that $\frac{BK}{CK} = \frac{3}{4}$. What is the area of triangle $BMK$?

3. Let there be given a triangle $ABC$ and points $M$ and $K$, on sides $AB$ and $BC$ of this triangle, respectively, such that $\frac{BM}{AB} = \frac{3}{7}$ and $\frac{BK}{CK} = \frac{2}{9}$. It is known that the area of triangle $ABC$ is equal to $A$. Find the area of triangle $BMK$.

4. Given a triangle $ABC$, let $M$ be a point on the straight line $\overrightarrow{AB}$ such that $A$ lies between $M$ and $B$, and such that $\frac{BM}{AB} = \frac{9}{5}$. Let $K$ be a point on side $BC$, such that $\frac{BK}{CK} = \frac{4}{7}$. It is known that the area of triangle $ABC$ is equal to $A$. Find the area of triangle $BMK$.

The first of these problems is essentially a review — the students by this time have usually already discussed the fact that, for example, a median divides a triangle into two triangles of equal area, since the heights of the two obtained triangles are the same as the height of the original triangle, while their bases are twice as small. Consequently, in the problem posed above, it is not difficult to find that the area of the obtained triangle is three times smaller than the area of the given triangle. The second problem is analogous in principle, but involves a new step — the argument just made must be applied for a second time to the new triangle. The third problem combines what was done in the first and second problems, but now the students must themselves break the problem down into separate parts, i.e. to make an additional construction. Moreover, the numbers given are somewhat more complicated than the numbers in the preceding problems. The fourth problem is identical to the third in every respect except that the positions of the points $A$, $B$, and $M$ are somewhat different — in other words, the diagram will have a somewhat different appearance [Figs. 1(a) and 1(b)].
In this way, the whole idea of the theorem’s proof is discussed. What is required to complete the proof of the theorem? It is still necessary to make a transition from expressing the idea in terms of numerical values to expressing it in terms of general relations. The expression “find the area of the obtained triangle, based on your knowledge of the area of the given triangle” must be replaced with an expression about relations between areas (which will be natural, since it is already clear why this relation is needed). Finally, it is necessary to examine the general case, where two different triangles with congruent angles are given, rather than two triangles with a common angle, i.e. it must be shown that the general case can be reduced to the case that has been investigated, by “superimposing” one triangle on the other. All of this can usually be
done by the students themselves, i.e. they can be told to carry out the proof of the theorem as a final problem. But even if teachers decide that it would be better if they themselves sum up the discussion and draw the necessary conclusions, the students will be prepared.

It must be pointed out here that genuine problem solving is often too categorically contrasted with the solving of routine exercises. The implication thus made is that in order to involve students in authentic problem solving in class, they must be presented with a situation that is altogether unfamiliar to them. Furthermore, because it is in reality clear to everyone that nothing good can come of such an exercise in the classroom, students are in fact not given difficult and unfamiliar problems. Instead, they receive either mere rhetoric or else long problems or word problems in place of substantive problems.

The whole difference between solving problems in class and solving problems chosen at random at home lies in the fact that in class the teacher can help — not by giving direct hints, but by organizing the problem set in a meaningful way. Indeed, even problems that seem absolutely analogous (such as problems 1 and 2 above) in reality demand a certain degree of creativity and cannot be considered to be based entirely on memory; this has been discussed, for example, by the Russian psychologist Kalmykova (1981). A structured system of problems enables students to solve problems that are challenging in the full sense of the word. Yes, the teacher helps them by breaking down a difficult problem into problems they are capable of solving, but precisely as a result of this the students themselves learn that problems may be broken down in this way and thus become capable of similarly breaking down problems on their own in the future. This is precisely the kind of scaffolding which enables students to accomplish what they cannot yet do on their own, as described by Vygotsky (1986).

It is important to emphasize that the program in mathematics has been constructed and remains constructed (even now, despite reductions in the amount of time allocated for mathematics and increases in the quantity of material studied) in such a way that it leaves class time not only for introducing one or another concept, but also for working with it. Consequently, even in lessons which would be classified as lessons devoted to reinforcing what has already been learned (according to the classification system discussed above),
students not only review what they have learned, but also discover new sides of this material. To illustrate, let us briefly describe a seventh-grade lesson on “Polynomials,” which follows a section on the formulas for the squares of the sums and differences of expressions.

At the beginning of the lesson, the teacher conducts a “dictation”: she dictates several expressions, such as “the square of the sum of the number $a$ and twice the number $b$” or “the square of the difference of three times the number $c$ and half of the number $d$.” The class, as well as two students called up to the blackboards, write down the corresponding algebraic expressions and, manipulating them in accordance with the formulas, put them into standard form. The blackboards are positioned in such a way that the work of the students at the blackboards cannot be seen by the rest of the class. Once they complete the dictation, students in neighboring seats switch notebooks, the class turns to face the blackboards, and all the students together check the results, discussing any mistakes that have been made (students in neighboring seats check one another’s work).

Then the class is given several oral problems in a row, which have also been written down on the blackboard, and which require the students to carry out computations. Without writing anything down, the students determine each answer in their minds and raise their hands. When enough hands are raised, the teacher asks several students to give the answer and explain how it was obtained. The problems given include the following:

1. $21^2 + 2 \cdot 21 \cdot 9 + 9^2$
2. $2009^2 + 2010^2 - 4020 \cdot 2009$
3. $(100 + 350)^2 - 100^2 - 350^2$
4. $17^2 + 2 \cdot 17 \cdot 13 + 13^2$
5. $32^2 - 2 \cdot 32 \cdot 12 + 12^2$

In a final problem, the teacher deliberately writes down one number illegibly (it is denoted as $\otimes$): $\frac{50^2 - \otimes + 30^2}{13^2 + 2 \cdot 13 \cdot 7 + 49}$. The students are then asked what number should be written down in order to make this expression analogous to the previous one.

After solving and discussing these problems, the students are asked to solve several problems involving simplifications and transformations. The students work in their notebooks. In conclusion, students
are called up to the blackboards to write down the answers to these problems, one by one, along with necessary explanations. The problems given include the following:

1. Write each of the following expressions in the form of a square of a binomial, if possible: (a) $x^2 + 16 - 8x$; (b) $4r^2 + 12r + 9$.

2. Find a number $k$ such that the following expression becomes the square of a binomial: $z^2 + 8z + k$.

3. Simplify the following expressions: (a) $a^2 - 2a + 1 - (a + 1)^2$;
   (b) $2m^2 - 12m + 18 - (3 - m)^2$;
   (c) $(m - 8)^2 - (m - 10)(m - 6)$;
   (d) $\frac{(x + 2)^2 - 4(x + 2) + 4}{(x + 4)^2}$.

Subsequently, the teacher inquires about deriving the formula for the square of the sum of a trinomial and asks the students to discuss the following, allegedly correct formula:

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 3ac + 4bc.$$  

After the students discuss this formula, they are asked to derive the correct formula on their own (the result is written down on the blackboard).

The lesson concludes with the students being asked to prove that, for any natural values of $n$, the expression $9n^2 - (3n - 2)^2$ is divisible by 4 (more precisely, this problem is given to those students who have already completed the previous problem).

As we can see, it would be somewhat naive to attempt to describe this lesson without taking into account the specific problems that were given to the students. Collective work alternates with individual work here, and written work alternates with oral work. The teacher, even when using the rather limited amount of material available to seventh graders, tries to teach them not a formula, but the subject itself. For this reason, connections are constantly made with various areas of mathematics and various methods of mathematics — the students communicate mathematically, make computations, carry out proofs, evaluate, check the justifiability of a hypothesis, and construct a problem on their own (even if relying on a model). They apply what they have learned, both while carrying out computations and, for example, while proving the last proposition concerning divisibility, but they also derive new facts (such as a new formula).
On the one hand, nearly all of the problems are different; no problems are different only by virtue of using different numbers and are otherwise identical. On the other hand, the problems given to the students echo one another and, to some extent, build on one another. For example, in the computational exercise No. 1, the formula is applied in standard form, while in No. 2 a certain rearrangement must be made. Problems involving the simplification of algebraic expressions recall the computational problems given earlier. The problem in which students are asked to determine a $k$ to obtain the square of a binomial has something in common with the problem containing the illegible number, and so on — not to mention that the formulas repeated during the first stage become the foundation for all that follows.

The lesson is structured rather rigidly in the sense indicated above, i.e. in terms of the presence of links and connections that make the order of the problems far from arbitrary. At the same time, a lesson with such content requires considerable flexibility and openness on the teacher’s part. For example, the hypothesis that $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 3ac + 4bc$ may be rejected by the students for various reasons — say, because the expression proposed is not symmetric (the students will most likely express this thought in their own way, and the teacher will have to work to clarify it), or simply because when certain numbers are substituted for the variables, e.g. $a = b = c = 1$, the two sides of the equation are not equal. However, the students might also express opinions that they cannot convincingly justify (for example, that the coefficients cannot be 3 and 4 because the formulas studied previously did not contain these coefficients). The teacher must have the ability both to get to the bottom of what students are trying to say in often unclear ways, and to take a proposition and quickly show its author and the whole class that it is open to question and has not been proven.

One of the authors of this chapter (Karp, 2004) has already written elsewhere about the complex interaction between the mathematical content and the pedagogical form of a lesson. Sometimes the teacher is able to achieve an interaction between content and form that has an emotional effect on the students comparable to the effect made by works of art. However, even given the seemingly simple
composition of the lesson examined above, the choice of adequate pedagogical techniques for the lesson is essential. It is difficult for students in general and for seventh graders in particular to remain at the same level of concentration for the entire 45 minutes (without suggesting that issues of discipline can always be resolved and only through successful lesson construction, we will nonetheless say that it would be ill-advised to expect 13-year-old children to sit quietly and silently during a lesson in which they have nothing to do or, on the contrary, are given assignments that are too difficult for them). Consequently, questions arise about how more and less intensive parts of a lesson can alternate with one another, and about the rhythm and tempo of the lesson in general. In the lesson examined above, a period of intense concentration (dictation) was followed by a less intense period, during which the students’ work was checked; intense oral work was followed by more peaceful written work. Consequently, collective work was followed by individual work, with students working at their own individual speeds. At this time, the teacher could adopt a more differentiating approach, perhaps even giving some students problems different from those being solved by the whole class. An experienced teacher almost automatically identifies such periods of differing intensities during a lesson and selects problems accordingly.

Note that group work, which has become more popular in recent years partly because of the influence of Western methodology, is still (as far as can be judged) rarely employed; this contrasts with working in pairs, including checking answers in pairs, as exemplified in the lesson examined earlier. Without entering into a discussion on the advantages and disadvantages of working in groups, and without examining the difficulties connected with frequently employing this approach, we should say that this approach has not been traditionally used in Russia (as we noted, even the classroom desks are arranged in such a way that it is difficult to organize group work). On the other hand, administrative fiat in Russia and the USSR has imposed so many methodologies which were declared to be the only right methodologies that Russian teachers usually react skeptically to methodologies that are too vehemently promoted. The creation of a problem book for group work presents an
interesting methodological problem, i.e. the creation of a collection of substantive problems in school materials for the solving of which group effort would be genuinely useful, so that working in groups would not simply involve students comparing and coordinating answers or strong students giving solutions to weaker ones. As far as we know, no such problem book has yet been published in Russia.

It should not be supposed, of course, that every lesson must be constructed as a complicated alternation of various pedagogical and methodological techniques. Mathematics studies in general and mathematics lessons in particular can to some degree consist of monotonous independent work involving the systematic solving of problems. The difference between this kind of work and completely independent work on problems chosen “at random” lies in the fact that the problems in the former case are selected according to some thematic principle or because the solutions involve the same technique and so enable the students to better grasp the material. As an example, let us examine part of a problem set from a course in geometry for a class that is continuing to study relations between the areas of triangles with congruent angles, which we mentioned earlier:

1. Points $M$ and $N$ lie on sides $AB$ and $BC$, respectively, of triangle $ABC$. $\frac{AM}{MB} = \frac{5}{3}$; $\frac{BN}{NC} = \frac{7}{8}$. Find: (a) the ratio of the area of triangle $BMN$ to the area of triangle $ABC$; (b) the ratio of area of quadrilateral $AMNC$ to the area of triangle $BMN$.

2. Triangle $ABC$ is given. Point $A$ divides segment $BK$ into two parts such that the ratio of the length of $BA$ to that of $AK$ is 3:2. Point $F$ divides segment $BC$ into two parts such that the ratio of their lengths is 5:3. The area of triangle $BKF$ is equal to 2. Find the area of triangle $ABC$.

3. The vertices of triangle $MNK$ lie on sides $AB$, $BC$, and $AC$, respectively, of triangle $ABC$ in such a way that $AM:MB = 3:2$; $BN = 6NC$; and $K$ is the midpoint of $AC$. Find the area of triangle $MNK$ if the area of triangle $ABC$ is equal to 70.

4. $ABCD$ is a parallelogram. Point $F$ lies on side $BC$ in such a way that $BF:FC = 5:2$. Point $Q$ lies on side $AB$ in such a way that $AQ = 1.4QB$. Find the ratio of the area of parallelogram $ABCD$ to the area of triangle $DFQ$.
As we can see, the set begins with a problem that the students already know. They can now solve it by directly applying the theorem to two triangles that have the common angle B. In problems 1(b) and 2, the application of the theorem becomes somewhat less straightforward — in problem 1(b), the students have to see that the area of the quadrilateral AMNC and the area of the triangle BMN together make up the area of the triangle ABC, while in problem 2 they must find the area of the given triangle ABC rather than of the obtained triangle, as was the case earlier. In problem 3, the basic idea has to be applied several times. In problem 4, this must be done in a parallelogram, which must additionally be broken down into triangles. Such a problem set can be expanded with more difficult problems.

Note that such problems may be used in another class: with eleventh graders when reviewing plane geometry. In that case, it would be natural to continue the series using analogous problems connected with the volumes of tetrahedra and based on the following proposition:

If a trihedral angle in tetrahedron $ABCD$ is congruent to a trihedral angle in tetrahedron $A_1B_1C_1D_1$ then the volumes of the two tetrahedra stand in the same ratio to each other as the lengths of the sides that form this angle. For example, let the trihedral angle $ABCD$ be congruent to the trihedral angle $A_1B_1C_1D_1$. Then

$$\frac{V_{ABCD}}{V_{A_1B_1C_1D_1}} = \frac{AB \cdot AC \cdot AD}{A_1B_1 \cdot A_1C_1 \cdot A_1D_1}$$

(where $V$ represents volume).

Until now, we have emphasized the importance of problem sets. But sometimes it makes sense to construct a lesson around a single problem. For example, the outstanding St. Petersburg teacher A. R. Maizelis (2007) would ask his class to find as many solutions as they could to the following problem:

Given an angle $ABC$ and a point $M$ inside it, draw a segment $CD$ such that its endpoints are on the sides of the angle and the point $M$ is its midpoint.

Students would offer many different solutions (usually in one way or another involving the construction of a parallelogram whose diagonals intersected at the point $M$). After this, the same kind of problem was posed, not about an angle and a point $M$ but, say, about a straight
line, a circle, and a point \( M \). Usually, none of the previously offered solutions could be transferred to this new problem, which nevertheless became easy to solve once the students realized that they had to make use of the properties of central symmetry. Such a lesson expanded their understanding of the meaning of the theorems they had learned earlier; it helped them evaluate the possibilities of applying transformations; and, more broadly, it trained them to become genuine problem solvers who discovered aesthetic pleasure from the actual process of solving problems.

5  Epilogue: Bad Lessons, and What One Would Like to Hope for

Above, we spoke mainly about “good” lessons. It would be misleading, of course, to claim that all lessons in Russia could be so characterized. Paraphrasing Leo Tolstoy’s famous line about unhappy families, one could say that every bad lesson is bad in its own way. The system of rigid monitoring and uniform rigid requirements is a thing of the past. Searching for a general formula for failure, and thus for a general prescription for turning bad lessons into good ones, is futile. Yet, certain patterns can still be identified.

The system of intensive work and high demands in class, described above, presupposed systematic work outside of class as well. In the 1930s, and indeed much later also, teachers were explicitly required, in addition to normal classroom lessons, to conduct additional lessons with weak students. These lessons, in turn, were not always successful; often enough, they consisted of “squeezing out” a positive grade. Yet their very existence (even as a form of punishment for students who had failed to fulfill what was required of them in time and were for this reason forced to spend time after school) played a definite role. Nor did it occur to anyone to pay teachers extra wages for such classes — these were considered a part of ordinary work. On the other hand, from a certain point on, a well-developed system for working with the strongest students existed. In addition to the fact that the strongest students would leave ordinary schools to attend schools with an advanced course in mathematics, there existed mathematics
clubs or “math circles” (including clubs in schools), optional classes, regularly assigned difficult problems, and the like, which to some extent facilitated the productive engagement of the strongest students as well.

In using the past tense, we do not wish to imply that working with students outside class boundaries has now receded into the past. This form of work still exists, although — probably above all for economic reasons — not everywhere. Meanwhile, unoccupied students and students who have not been given enough to do invariably create problems during the lesson. To repeat, this is not the only cause of poor discipline in some schools, which affects all classes, particularly mathematics classes. While it is impossible to resolve social problems by relying exclusively on a teacher’s skills, the absence of such skills may exacerbate such problems and give rise to discipline problems where no deep social reasons for them exist.

As one negative development of recent times, we should mention a specific change in the attitude of some teachers. It is fitting to criticize the perfunctory optimism of Soviet era schools, with their cheerful slogans such as: “If you can’t do it, we’ll teach you how to do it, and if you don’t want to do it, we’ll force you to do it!” Nevertheless, the system as a whole encouraged teachers to believe that practically all students must be raised to a certain level (even if this was not always possible in practice). Everyone recognized the importance of mathematics education in this context. Posters with the words of the great Russian scientist Mikhail Lomonosov were hung (and have hung to this day) in virtually every mathematics classroom: “Mathematics must be studied if for no other reason than because it sets the mind in order.” Who would argue that the mind should not be set in order? Again, the authors of this chapter would like to believe that society as a whole has largely preserved its respect for the study of mathematics and that this gives reason to hope that current difficulties will be overcome. Yet in all fairness it must also be pointed out that the justifiable fight against a fixation on universal academic advancement has sometimes turned into an unwillingness to try to teach students (we should qualify this statement at once, by saying that it is based on observations, not on systematic studies or statistics — we have no such data at our disposal).
In classes, this is evidenced by the fact that even those teachers who adhere to the form of the intensive lesson are not concerned about its results. Probably the worst class ever observed by one of the authors of this chapter was a class that he visited once while supervising a school for juvenile delinquents. The problem had nothing to do with discipline, as might have been expected. On the contrary, the discipline was excellent, and the teacher began by energetically conducting a mathematical dictation; he then explained new material, making use of a variety of techniques, and this was followed by independent work and a mathematical game — and so on and so forth. The trouble with this display of pedagogical and methodological fireworks was that the material being studied was eighth-grade material, while all of the students — as was obvious from their answers — barely knew mathematics at a fourth-grade level. A strange exercise was taking place during which no one learned anything. The teacher, however, was not in the least disconcerted by the students’ absurd answers — the class, as it were, had a legitimate right to be considered weak.

This example is to some extent exceptional, but the absence of a goal truly to teach students and the willingness to ignore reality may be the most important reasons for bad classes, i.e. classes that fail to teach students, in ordinary schools. Indeed, its manifestations may be observed in selective schools as well, when teachers set goals they know are unrealizable, such as attempting to cram into a single lesson material that would be challenging to cover in three lessons — since, after all, the children are good students. However paradoxical it may seem, Russian respect for mathematics sometimes has negative consequences in such cases: both the children and their parents make the mistake of thinking that a large quantity of work implies a high quality of education.

The art of being a teacher in any country, including Russia, presupposes the ability to choose problems and to leave enough time for their solution, to determine what will be tiring for the students and what will give them a chance to rest. It presupposes the ability to know a large number of useful sources and to pose the right questions on the spot in the classroom, displaying flexibility and departing from what was previously planned. And the list goes on. It is not difficult to
provide examples of Russian classes in which the teachers themselves did not really know the subject and thus could not teach their students, or in which the predetermined lesson plan collapsed because the very first activity consumed all of the class time, making the activity pointless. Yet, a teacher’s inability to plan adequate time for a lesson and even insufficient knowledge of the subject may usually be overcome through systematic and persistent work — and, above all, through a commitment to overcoming these weaknesses.

The traditions of Russian mathematics education, including those of conducting and constructing lessons, took shape as part of the complex and often frightening development of Russian history. People sometimes became teachers of mathematics who, under different circumstances, might have been department chairs at leading universities. The rigid and merciless system forced teachers to work long and hard, usually without minimally adequate compensation. The system raised mathematics to a privileged position, while often at the same time destroying existing scholarly traditions and instruction of the humanities. This same system gave rise to a meaningless formalism in the teaching of mathematics and to a fear of deviating from approved templates.

Nevertheless, over the course of a complicated development in a country that possesses enormous human and cultural resources, traditions of intensive, genuine, and fully instructive mathematics education emerged. Regardless of the circumstances that brought these wonderful teachers to schools, these individuals created models which all teachers to this day can aspire to match. These are models of an attitude toward one’s work and its inherent problems, models of relations with students, models of lessons taught. These models do not concern the details, which inevitably change and are renewed over time, as the authors of this chapter witnessed when certain topics that had previously been deemed important were dropped from the curriculum; even less do they concern technological implements, such as the slide rule. At stake, rather, are models of how to achieve the goal of genuinely teaching and developing children during every class, and models of how to employ a rich palette of techniques, methods, and problems for doing so. These models continue to exert their influence — they
have been seen by thousands of people, including those who became teachers and those who became parents, and who want their children to have education similar to what they once had. It is important to remember that these models have to a certain extent been reflected in textbooks and problem books, which in their turn oriented and educated new teachers. It is on the vitality of existing traditions that one would like to pin one’s hopes.

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