Spectral Theory of Pseudo-Differential Operators on $S^1$

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Abstract. For a bounded pseudo-differential operator with the dense domain $C^\infty(S^1)$ on $L^p(S^1)$, the minimal and maximal operator are introduced. An analogue of Agmon-Douglis-Nirenberg [1] is proved and then is used to prove the uniqueness of the closed extension of an elliptic pseudo-differential operator of symbol of positive order. We show the Fredholmness of the minimal operator. The essential spectra of pseudo-differential operators on $S^1$ are described.

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1. Introduction

In this paper the focus is on pseudo-differential operators on the unit circle $S^1$ centered at the origin. For $-\infty < m < \infty$, let $S^m(S^1 \times \mathbb{Z})$ be the set all functions $\sigma$ in $C^\infty(S^1 \times \mathbb{Z})$ such that for all nonnegative integers $\alpha$ and $\beta$ there exists a positive constant $C_{\alpha,\beta}$ for which

$$|((\partial_\theta^\alpha \partial_n^\beta \sigma)(\theta, n))| \leq C_{\alpha,\beta} (1 + |n|)^{m-\beta}, \quad \theta \in [-\pi, \pi], \quad n \in \mathbb{Z}.$$  

Let $\sigma \in S^m(S^1 \times \mathbb{Z})$, $-\infty < m < \infty$. Then we define the pseudo-differential operator $T_\sigma$ on $L^1(S^1)$ by

$$(T_\sigma f)(\theta) = \sum_{n \in \mathbb{Z}} e^{in\theta} \sigma(\theta, n)(\mathcal{F}_{S^1} f)(n), \quad \theta \in [-\pi, \pi],$$

where

$$(\mathcal{F}_{S^1} f)(n) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta) \, d\theta, \quad n \in \mathbb{Z}.$$  

Basic properties of pseudo-differential operators with symbols in $S^m(S^1 \times \mathbb{Z})$, $-\infty < m < \infty$, can be found in [2, 3, 4, 6, 10, 9]. The basic calculi for the
product and the formal adjoint of pseudo-differential operators with symbols in $S^m(S^1 \times \mathbb{Z})$ can be found in [9].

A symbol $\sigma$ in $S^m(S^1 \times \mathbb{Z})$, $-\infty < m < \infty$, is said to be elliptic if there exist positive constants $C$ and $R$ such that

$$|\sigma(\theta, n)| \geq C(1 + |n|)^m, \quad |n| \geq R, \quad \theta \in [-\pi, \pi].$$

The following theorem gives a parametrix for an elliptic pseudo-differential operator with symbol in $S^m(S^1 \times \mathbb{Z})$, $\infty < m < -\infty$, see [9].

**Theorem 1.1.** Let $\sigma \in S^m(S^1 \times \mathbb{Z})$, $-\infty < m < \infty$ be elliptic. Then there exists a symbol $\tau \in S^{-m}(S^1 \times \mathbb{Z})$ such that $T_\sigma T_\tau = I + K$ and $T_\tau T_\sigma = I + R$, where $K$ and $R$ are infinitely smoothing in the sense that they are pseudo-differential operators with symbols in $\cap_{m \in \mathbb{R}} S^m(S^1 \times \mathbb{Z})$.

Similar results for the symbol class $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ of the pseudo-differential operators on $\mathbb{R}^n$ have been studied for example in [15].

In Section 2, we recall $L^p$-Sobolev spaces $H^{s,p}$, $-\infty < s < \infty$, $1 \leq p \leq \infty$, and we give some of the results in [7]. Then in Section 3, we consider bounded pseudo-differential operators $T_\sigma$ on $L^p(S^1)$, $1 < p < \infty$ with dense domain $C^\infty(S^1)$. The smallest and largest closed extension of $T_\sigma$ are provided. The analogue of Agmon-Douglis-Nirenberg [1], is given to prove that for an elliptic symbol $\sigma$ of positive order $m$, the corresponding pseudo-differential operator has a unique closed extension with domain $H^{m,p}$ on $L^p(S^1)$. In Section 4, we focus on Fredholmness of pseudo-differential operators and its essential spectrum. Results on the Fredholmness of pseudo-differential operators on $\mathbb{R}^n$ can be found in [16, 13]. By using Theorem 2.9 in [7], we see that the minimal operator of an elliptic pseudo-differential operator of positive order is Fredholm. The essential spectra of the pseudo-differential operator and the minimal (maximal) operator are then provided. Similar results for the SG Pseudo-differential operator on $\mathbb{R}^n$ are given in [5, 8].

2. $L^p$-Sobolev spaces

For $-\infty < s < \infty$, let $J_s$ be the pseudo-differential operator with symbol $\sigma_s$ given by

$$\sigma_s(n) = (1 + |n|^2)^{-s/2}, \quad n \in \mathbb{Z}.$$

$J_s$ is called the Bessel potential of order $s$.

Now, for $-\infty < s < \infty$ and $1 \leq p \leq \infty$, we define the $L^p$-Sobolev space $H^{s,p}$ to be the set of all tempered distributions $u$ for which $J_{-s}u$ is a function in $L^p(S^1)$. Then $H^{s,p}$ is a Banach space in which the norm $\| \cdot \|_{s,p}$ is given by

$$\|u\|_{s,p} = \|J_{-s}u\|_{L^p(S^1)}, \quad u \in H^{s,p}.$$

It is easy to show that for $-\infty < s, t < \infty$, $J_t$ is an isometry of $H^{s,p}$ onto $H^{s+t,p}$. 
The following theorem is known as Sobolev embedding theorem.

**Theorem 2.1.** Let $1 < p < \infty$ and $s \leq t$. Then $H^{t,p} \subseteq H^{s,p}$ and

$$\|u\|_{s,p} \leq \|u\|_{t,p}, \quad u \in H^{t,p}.$$ 

**Proposition 2.2.** Let $\sigma \in S^{m}(S^{1} \times \mathbb{Z})$, $-\infty < m < \infty$. Then $T_{\sigma} : H^{s,p} \rightarrow H^{s-m,p}$ is a bounded linear operator for $1 < p < \infty$.

**Proposition 2.3.** Let $s < t$. Then the inclusion operator $i : H^{t,p} \hookrightarrow H^{s,p}$ is compact for $1 \leq p \leq \infty$.

The results above can be found in [7].

### 3. Minimal and maximal operators

Let $\sigma \in S^{m}(S^{1} \times \mathbb{Z})$, $m \in \mathbb{R}$. Then the formal adjoint of $T_{\sigma}$, denoted $T_{\sigma}^{*}$ is a linear operator on $C^{\infty}(S^{1})$ such that

$$(T_{\sigma} \varphi, \psi) = (\varphi, T_{\sigma}^{*} \psi), \quad \varphi, \psi \in C^{\infty}(S^{1}).$$

It can be proved that the formal adjoint of $T_{\sigma}$ is a pseudo-differential operator of symbol of order $-m$ (see [10]). The following proposition guarantee that the minimal operator of $T_{\sigma}$ exists.

**Proposition 3.1.** Let $S^{m}(S^{1} \times \mathbb{Z})$, $-\infty < m < \infty$. Then $T_{\sigma} : L^{p}(S^{1}) \rightarrow L^{p}(S^{1})$ is closable with dense domain $C^{\infty}(S^{1})$ for $1 < p < \infty$.

**Proof.** Let $\{\varphi_{k}\}_{k=1}^{\infty}$ be a sequence in $C^{\infty}(S^{1})$ such that $\varphi_{k} \rightarrow 0$ and $T_{\sigma} \varphi_{k} \rightarrow f$ for some $f$ in $L^{p}(S^{1})$ as $k \rightarrow \infty$. We only need to show that $f = 0$. We have

$$(T_{\sigma} \varphi_{k}, \psi) = (\varphi_{k}, T_{\sigma}^{*} \psi), \quad \psi \in C^{\infty}(S^{1}), \quad k = 1, 2, \ldots.$$ 

Let $k \rightarrow \infty$, then $(f, \psi) = 0$ for all $\psi \in C^{\infty}(S^{1})$. By the density of $C^{\infty}(S^{1})$ in $L^{p}(S^{1})$, it follows that $f = 0$. \qed

Consider $T_{\sigma} : L^{p}(S^{1}) \rightarrow L^{p}(S^{1})$ with domain $C^{\infty}(S^{1})$. Then by Proposition 3.1, $T_{\sigma}$ has a closed extension. Let $T_{\sigma,0}$ be the minimal operator of $T_{\sigma}$ which is the smallest closed extension of $T_{\sigma}$. Then the domain $D(T_{\sigma,0})$ of $T_{\sigma,0}$ consists of all functions $u \in L^{p}(S^{1})$ for which there exists a sequence $\{\varphi_{k}\}_{k=1}^{\infty}$ in $C^{\infty}(S^{1})$ such that $\varphi_{k} \rightarrow u$ in $L^{p}(S^{1})$ and $T_{\sigma} \varphi_{k} \rightarrow f$ for some $f \in L^{p}(S^{1})$ in $L^{p}(S^{1})$ as $k \rightarrow \infty$. It can be shown that $f$ does not depend on the choice of $\{\varphi_{k}\}_{k=1}^{\infty}$ in $C^{\infty}(S^{1})$ and $T_{\sigma,0} u = f$.

We define the linear operator $T_{\sigma,1}$ on $L^{p}(S^{1})$ with domain $D(T_{\sigma,1})$ by the following. Let $f$ and $u$ be in $L^{p}(S^{1})$. Then we say that $u \in D(T_{\sigma,1})$ and $T_{\sigma,1} u = f$ if and only if

$$(u, T_{\sigma}^{*} \varphi) = (f, \varphi), \quad \varphi \in C^{\infty}(S^{1}).$$

It can be proved that $T_{\sigma,1}$ is a closed linear operator from $L^{p}(S^{1})$ into $L^{p}(S^{1})$ with domain $D(T_{\sigma,1})$ containing $C^{\infty}(S^{1})$. In fact, $C^{\infty}(S^{1})$ is contained in the domain $D(T_{\sigma,1}^{t})$ of the true adjoint $T_{\sigma,1}^{t}$ of $T_{\sigma,1}$. Furthermore, $T_{\sigma,1}(u) = T_{\sigma}(u)$ for all $u$ in $D(T_{\sigma,1})$. 

It is easy to see that $T_{\sigma,1}$ is an extension of $T_{\sigma,0}$. In fact $T_{\sigma,1}$ is the largest closed extension of $T_{\sigma}$ in the sense that if $B$ is any closed extension of $T_{\sigma}$ such that $C^\infty(\mathbb{S}^1) \subseteq D(B')$, then $T_{\sigma,1}$ is an extension of $B$. $T_{\sigma,1}$ is called the maximal operator of $T_{\sigma}$. The following theorem is an analogue of Agmon-Douglis-Nirenberg in [1].

**Proposition 3.2.** Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m > 0$ be elliptic. Then there exist positive constants $C$ and $D > 0$ such that

$$C\|u\|_{m,p} \leq \|T_{\sigma}u\|_{L^p(\mathbb{S}^1)} + \|u\|_{L^p(\mathbb{S}^1)} \leq D\|u\|_{m,p}, \quad u \in H^{m,p}.$$  

**Proof.** By the boundedness of $T_{\sigma}$ in Proposition 2.2 and the boundedness of the inclusion operator in Theorem 2.1, there exists a positive constant $D$ such that for all $u \in H^{m,p}$,

$$\|T_{\sigma}u\|_{L^p(\mathbb{S}^1)} + \|u\|_{L^p(\mathbb{S}^1)} \leq D\|u\|_{m,p}, \quad u \in H^{m,p}.$$  

Since $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ is elliptic, by Theorem 1.1, there exists a symbol $\tau \in S^{-m}(\mathbb{S}^1 \times \mathbb{Z})$ such that

$$u = T_{\tau}T_{\sigma}u - Ru, \quad u \in H^{m,p},$$

where $R$ is an infinitely smoothing operator in the sense that $R$ is a pseudodifferential operator with symbol in $\cap_{m \in \mathbb{R}} S^m(\mathbb{S}^1 \times \mathbb{Z})$. By using Proposition 2.2 again, $T_{\sigma}u \in L^p(\mathbb{S}^1)$. Therefore, $T_{\tau}T_{\sigma}u \in H^{m,p}$, for all $u \in H^{m,p}$, Moreover there exists a positive constant $C$ such that

$$\|u\|_{m,p} \leq C(\|T_{\sigma}u\|_{L^p(\mathbb{S}^1)} + \|u\|_{L^p(\mathbb{S}^1)}), \quad u \in H^{m,p}. \quad \Box$$

We have the following result which we use in the next theorem.

**Lemma 3.3.** Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then $C^\infty(\mathbb{S}^1)$ is dense in $H^{s,p}$.

**Proof.** Let $u \in H^{s,p}$. Then $J_{-s}u \in L^p(\mathbb{S}^1)$. Since $C^\infty(\mathbb{S}^1)$ is dense in $L^p(\mathbb{S}^1)$, there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{S}^1)$ such that $\varphi_k \rightarrow J_{-s}u$ in $L^p(\mathbb{S}^1)$ as $k \rightarrow \infty$. Let $\psi_k = J_k\varphi_k$, $k = 1, 2, \ldots$. Then $\psi_k \in C^\infty(\mathbb{S}^1)$, $k = 1, 2, \ldots$, and

$$\|\psi_k - u\|_{s,p} = \|J_{-s}\psi_k - J_{-s}u\|_{L^p(\mathbb{S}^1)} = \|\varphi_k - J_{-s}u\|_{L^p(\mathbb{S}^1)} \rightarrow 0,$$

as $k \rightarrow \infty$, which completes the proof. \quad \Box

The following theorem gives the domain of the minimal operator of an elliptic pseudodifferential operator with symbol of positive order.

**Theorem 3.4.** Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m > 0$, be elliptic. Then $D(T_{\sigma,0}) = H^{m,p}$.

**Proof.** Let $u \in H^{m,p}$. Then by using the density of $C^\infty(\mathbb{S}^1)$ in $H^{m,p}$, there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{S}^1)$ such that $\varphi_k \rightarrow u$ in $H^{m,p}$ and therefore in $L^p(\mathbb{S}^1)$ as $k \rightarrow \infty$. By Proposition 3.2, $\varphi_k$ and $T_{\sigma}\varphi_k$ are Cauchy sequences in $L^p(\mathbb{S}^1)$. Therefore $\varphi_k \rightarrow u$ and $T_{\sigma}\varphi_k \rightarrow f$ for some $f \in L^p(\mathbb{S}^1)$ as $k \rightarrow \infty$. This implies that $u \in D(T_{\sigma,0})$ and $T_{\sigma,0}u = f$. Now assume that $u \in D(T_{\sigma,0})$. Then there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{S}^1)$ such that $\varphi_k \rightarrow u$ in $L^p(\mathbb{S}^1)$ and $T_{\sigma}\varphi_k \rightarrow f$, for some $f \in L^p(\mathbb{S}^1)$ as $k \rightarrow \infty$. So, by Proposition 3.2, $\{\varphi_k\}_{k=1}^\infty$ is a Cauchy sequence
in $H^{m,p}$. Since $H^{m,p}$ is complete, there exists $v \in H^{m,p}$ such that $\varphi_k \to v$ in $H^{m,p}$ as $k \to \infty$. By Sobolev embedding theorem $\varphi_k \to v$ in $L^p(S^1)$ which implies that $u = v \in H^{m,p}$.

The following theorem shows that the closed extension of an elliptic pseudo-differential operator on $L^p(S^1)$ with symbol $\sigma \in S^m(S^1 \times \mathbb{Z})$, $m > 0$, is unique and moreover by Theorem 3.4, its domain is $H^{m,p}$.

**Theorem 3.5.** Let $\sigma \in S^m(S^1 \times \mathbb{Z})$, $m > 0$, be elliptic. Then $T_{\sigma,0} = T_{\sigma,1}$.

**Proof.** Since $T_{\sigma,1}$ is a closed extension of $T_{\sigma,0}$, by Theorem 3.4, it is enough to show that $\mathcal{D}(T_{\sigma,1}) \subseteq H^{m,p}$. Let $u \in \mathcal{D}(T_{\sigma,1})$. By ellipticity of $\sigma$, there exists $\tau \in S^{-m}(S^1 \times \mathbb{Z})$ such that

$$u = T_\tau T_\sigma u - Ru,$$

where $R$ is an infinitely smoothing operator. Since $T_\sigma u = T_{\sigma,1} u \in L^p(S^1)$, by Proposition 2.2, it follows that $u \in H^{m,p}$, which completes the proof. □

4. Fredholm pseudo-differential operators

A closed linear operator $A$ from a complex Banach space $X$ into a complex Banach space $Y$ with dense domain $\mathcal{D}(A)$ is said to be Fredholm if

- the range of $A$, $R(A)$ is closed subspace of $Y$ and
- the null space of $A$, $N(A)$ and the null space of the true adjoint of $A$, $N(A^*)$ are finite dimensional.

The index of a Fredholm operator $A$ is defined by

$$i(A) = \dim N(A) - \dim N(A^*).$$

By Atkinson’s theorem, a closed linear operator $A : X \to Y$ with dense domain $\mathcal{D}(A)$ is Fredholm if and only if there exists a bounded linear operator $B : Y \to X$ such that $K_1 = AB - I : Y \to Y$ and $K_2 = BA - I : X \to X$ are compact operators.

Let $A : X \to X$ be a closed linear operator with dense domain $\mathcal{D}(A)$ in the complex Banach space $X$. Then the spectrum of $A$, $\Sigma(A)$ is defined by

$$\Sigma(A) = \mathbb{C} - \rho(A),$$

where $\rho(A)$ is the resolvent set of $A$ given by

$$\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is bijective}\}.$$ 

The essential spectrum $\Sigma_w(A)$ of $A$, which has been defined in [14] by Wolf given by

$$\Sigma_w(A) = \mathbb{C} - \Phi_w(A), \text{ where } \Phi_w(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is Fredholm}\}.$$ 

Note that $i(A - \lambda I)$ is constant for all $\lambda$ in a connected component of $\Phi_w(A)$.

The essential spectrum $\Sigma_s(A)$ of $A$ in sense of Schechter [11] is defined by

$$\Sigma_s(A) = \mathbb{C} - \Phi_s(A), \text{ where } \Phi_s(A) = \{\lambda \in \Phi_w(A) : i(A - \lambda I) = 0\}.$$
For the properties of essential spectra see [12]. The following theorem gives a sufficient condition for $T_\sigma : H^{s,p} \to H^{s-m,p}$ to be a Fredholm operator. The proof can be found in [7].

**Theorem 4.1.** Let $\sigma \in S^m(S^1 \times \mathbb{Z})$, $-\infty < m < \infty$ be elliptic. Then for all $-\infty < s < \infty$ and $1 < p < \infty$, $T_\sigma : H^{s,p} \to H^{s-m,p}$ is a Fredholm operator. In particular if $\sigma \in S^0(S^1 \times \mathbb{Z})$, then the bounded linear operator $T_\sigma : L^p(S^1) \to L^p(S^1)$ is Fredholm.

The following is an immediate corollary of Theorem 3.4 and Theorem 4.1.

**Corollary 4.2.** Let $\sigma \in S^m(S^1 \times \mathbb{Z})$, $m > 0$ be elliptic. Then for $1 < p < \infty$, $T_{\sigma,0}$ is a Fredholm operator on $L^p(S^1)$ with the domain $H^{m,p}$.

The following theorem gives the essential spectrum of an elliptic pseudo-differential operator of positive order.

**Theorem 4.3.** Let $\sigma \in S^m(S^1 \times \mathbb{Z})$, $m > 0$ be elliptic. Then

$$
\Sigma_w(T_{\sigma,0}) = \emptyset.
$$

**Proof.** Let $\lambda \in \mathbb{C}$. By Corollary 4.2, we need only to show that $\sigma - \lambda$ is elliptic. The ellipticity of $\sigma$, implies that there exist constants $C, R > 0$ such that

$$
|\sigma(\theta, n) - \lambda| \geq C(1 + |n|)^m - |\lambda| = (1 + |n|)^m(C - \frac{|\lambda|}{1 + |n|}), \quad \theta \in [-\pi, \pi],
$$

whenever $|n| \geq R$. Since $(1 + |n|)^m \to \infty$ as $|n| \to \infty$, there exists $M > 0$ such that

$$
|\sigma(\theta, n) - \lambda| \geq \frac{C}{2}(1 + |n|)^m, \quad |n| \geq M, \quad \theta \in [-\pi, \pi],
$$

which implies that $\sigma - \lambda$ is elliptic. $\square$

Let $\sigma \in S^m(S^1 \times \mathbb{Z})$, $m \geq 0$. Then the following theorem is a result on the essential spectra of the bounded pseudo-differential operator $T_\sigma$ with the domain $H^{m,p}$ on $L^p(S^1)$.

**Theorem 4.4.** Let $\sigma \in S^m(S^1 \times \mathbb{Z})$, $m \geq 0$. Then for $T_\sigma$ on $L^p(S^1)$ with the domain $H^{m,p}$, $1 < p < \infty$, we have

$$
\Sigma_w(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq L_i\},
$$

where

$$
L_i = \liminf_{|n| \to \infty}(\inf_{\theta \in [-\pi, \pi]}|\sigma(\theta, n)|)(1 + |n|)^{-m}.
$$

**Proof.** Let $\lambda \in \mathbb{C}$ be such that $|\lambda| < L_i$. Then there exists $\epsilon > 0$ such that

$$
|\lambda| + \epsilon < L_i.
$$

Since $m \geq 0$, it follows that $|\lambda| < (L_i - \epsilon)(1 + |n|)^m$. On the other hand, there exists a positive constant $R$ such that

$$
\inf_{|n| \geq R}(\inf_{\theta \in [-\pi, \pi]}|\sigma(n, \theta)|)(1 + |n|)^{-m} > L_i - \frac{\epsilon}{2}.
$$
So, for $|n| \geq R$,

$$|\sigma(\theta, n) - \lambda| \geq |\sigma(\theta, n)| - |\lambda|$$

$$> (L_i - \frac{\epsilon}{2} - L_i + \epsilon)(1 + |n|)^m$$

$$= \frac{\epsilon}{2}(1 + |n|)^m, \quad \theta \in [-\pi, \pi].$$

Therefore, $\sigma - \lambda$ is elliptic and hence $T_\sigma - \lambda I : L^p(S^1) \to L^p(S^1)$ with domain $H^{m,p}$ is Fredholm. Thus,

$$\{\lambda \in \mathbb{C} : |\lambda| < L_i\} \subseteq \Phi_w(T_\sigma),$$

which implies that

$$\Sigma_w(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq L_i\}. \quad \Box$$

We have the following theorem on the essential spectrum of a pseudo-differential operator of order 0 from $L^p(S^1)$ into $L^p(S^1)$.

**Theorem 4.5.** Let $\sigma \in S^0(S^1 \times \mathbb{Z})$. Then for $T_\sigma : L^p(S^1) \to L^p(S^1)$, $1 < p < \infty$, we have

$$\Sigma_s(T_\sigma) \subseteq \{\lambda : |\lambda| \leq L_s\},$$

where

$$L_s = \limsup_{|n| \to \infty} \sup_{\theta \in [-\pi, \pi]} |\sigma(\theta, n)|.$$  

**Proof.** Let $\lambda \in \mathbb{C}$ such that $|\lambda| > L_s$. Then there exists $\epsilon > 0$ such that

$$|\lambda| - \epsilon > L_s,$$

and there exists a positive number $R$ such that

$$\sup_{|n| \geq R} \sup_{\theta \in [-\pi, \pi]} |\sigma(\theta, n)| < L_s + \frac{\epsilon}{2}.$$  

For all $|n| \geq R$,

$$|\sigma(\theta, n) - \lambda| \geq |\lambda| - |\sigma(\theta, n)|$$

$$> L_s + \epsilon - L_s - \frac{\epsilon}{2}$$

$$= \frac{\epsilon}{2}, \quad \theta \in [-\pi, \pi].$$

Hence $\sigma - \lambda$ is elliptic and by Theorem 4.1, $T_\sigma - \lambda I : L^p(S^1) \to L^p(S^1)$ is Fredholm. Thus,

$$\{\lambda \in \mathbb{C} : |\lambda| > L_s\} \subseteq \Phi_w(T_\sigma),$$

which is the same as

$$\Sigma_w(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq L_s\}. $$

Since $\{\lambda \in \mathbb{C} : |\lambda| > L_s\}$ is a connected component of $\Phi_w(T_\sigma)$, it follows that $i(T_\sigma - \lambda I)$ is a constant for all $\lambda$ in $\{\lambda \in \mathbb{C} : |\lambda| > L_s\}$. On the other hand,

$$\rho(T_\sigma) \cap \{\lambda \in \mathbb{C} : |\lambda| > L_s\} \neq \emptyset.$$
Therefore, \( i(T_\sigma - \lambda I) = 0 \) for all \( \lambda \in \mathbb{C} : |\lambda| > L_s \). This implies that
\[
\Sigma_s(T_\sigma) \subseteq \{ \lambda \in \mathbb{C} : |\lambda| \leq L_s \}.
\]
\[\square\]
We have the following spectral alternative for a pseudo-differential operator with symbol in \( S^0(S^1 \times \mathbb{Z}) \).

**Corollary 4.6.** Let \( \sigma \in S^0(S^1 \times \mathbb{Z}) \) be such that
\[
\limsup_{|n| \to \infty} \sup_{\theta \in [-\pi, \pi]} |\sigma(\theta, n)| = \liminf_{|n| \to \infty} \inf_{\theta \in [-\pi, \pi]} |\sigma(\theta, n)| = L > 0.
\]
Then
\[
\Sigma_w(T_\sigma) = \{ \lambda \in \mathbb{C} : |\lambda| = L \} \quad \text{or} \quad \Sigma_s(T_\sigma) \subseteq \{ \lambda \in \mathbb{C} : |\lambda| = L \}.
\]

**Proof.** By Theorem 4.4 and Theorem 4.5,
\[
\Sigma_w(T_\sigma) \subseteq \{ \lambda \in \mathbb{C} : |\lambda| = L \}.
\]
Suppose that
\[
\Sigma_w(T_\sigma) \neq \{ \lambda \in \mathbb{C} : |\lambda| = L \}.
\]
Then there exists \( \lambda_0 \in \mathbb{C} \) such that \( |\lambda_0| = L \) and \( \lambda_0 \in \Phi_w(T_\sigma) \). On the other hand, by Theorem 4.5,
\[
\{ \lambda \in \mathbb{C} : |\lambda| > L \} \subseteq \Phi_s(T_\sigma).
\]
Hence using the fact that \( \Phi_w(T_\sigma) \) is an open set and the index of \( T_\sigma - \lambda I \) is constant on on every connected component of \( \Phi_w(T_\sigma) \) we get \( i(T_\sigma - \lambda I) = 0 \) for all \( \lambda \in \mathbb{C} \) with \( |\lambda| \neq L \), which is the same as
\[
\Sigma_s(T_\sigma) \subseteq \{ \lambda \in \mathbb{C} : |\lambda| = L \},
\]
as asserted. \[\square\]

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