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Introduction and overview

We first introduce the reader to the notions of a system node and an $L^p$-well-posed linear system with $1 \leq p \leq \infty$, and continue with an overview of the rest of the book.

1.1 Introduction

There are three common ways to describe a finite-dimensional linear time-invariant system in continuous time:

(i) the system can be described in the time domain as an input/output map $\mathcal{D}$ from an input signal $u$ into an output signal $y$;

(ii) the system can be described in the frequency domain by means of a transfer function $\hat{\mathcal{D}}$, i.e., if $\hat{u}$ and $\hat{y}$ are the Laplace transforms of the input $u$ respectively the output $y$, then $\hat{y} = \hat{\mathcal{D}} \hat{u}$ in some right half-plane;

(iii) the system can be described in state space form in terms of a set of first order linear differential equations (involving matrices $A$, $B$, $C$, and $D$ of appropriate sizes)

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t), \\
x(0) &= x_0.
\end{align*}$$

In (i)–(iii) the input signal $u$ takes its values in the input space $U$ and the output signal $y$ takes its values in the output space $Y$, both of which are finite-dimensional real or complex vector spaces (i.e., $\mathbb{R}^k$ or $\mathbb{C}^k$ for some $k = 1, 2, 3, \ldots$), and the state $x(t)$ in (iii) takes its values in the state space $X$ (another finite-dimensional vector space).

All of the three descriptions mentioned above are important, but we shall regard the third one, the state space description, as the most fundamental.
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one. From a state space description it is fairly easy to get both an input/output description and a transfer function description. The converse statement is more difficult (but equally important): to what extent is it true that an input/output description or a transfer function description can be converted into a state space description? (Various answers to this question will be given below.)

The same three types of descriptions are used for infinite-dimensional linear time-invariant systems in continuous time. The main difference is that we encounter certain technical difficulties which complicate the formulation. As a result, there is not just one general infinite-dimensional theory, but a collection of competing theories that partially overlap each other (and which become more or less equivalent when specialized to the finite-dimensional case). In this book we shall concentrate on two quite general settings: the case of a system which is either well-posed in an $L^p$-setting (for some $p \in [1, \infty]$) or (more generally), it has a differential description resembling (1.1.1), i.e., it is induced by a system node.

In order to give a definition of a system node we begin by combining the four matrices $A$, $B$, $C$, and $D$ into one single block matrix $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, which we call the node of the system, and rewrite (1.1.1) in the form

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0.$$  \hspace{1cm} (1.1.2)

For a moment, let us ignore the original matrices $A$, $B$, $C$, and $D$, and simply regard $S$ as a linear operator mapping $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$ into $\mathbb{R}^m$. If $U$, $X$, and $Y$ are all finite-dimensional, then $S$ is necessarily bounded, but this need not be true if $U$, $X$, or $Y$ is infinite-dimensional. The natural infinite-dimensional extension of (1.1.1) is to replace (1.1.1) by (1.1.2) and to allow $S$ to be an unbounded linear operator with some additional properties. These properties are chosen in such a way that (1.1.2) generates some reasonable family of trajectories, i.e., for some appropriate class of initial states $x_0 \in X$ and input functions $u$ the equation (1.1.2) should have a well-defined state trajectory $x(t)$ (defined for all $t \geq 0$) and a well-defined output function $y$. The set of additional properties that we shall use in this work is the following.

**Definition 1.1.1** We take $U$, $X$, and $Y$ to be Banach spaces (sometimes Hilbert spaces), and call $S$ a system node if it satisfies the following four conditions:

1. $S$ is a closed (possibly unbounded) operator mapping $\mathcal{D}(S) \subset \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$ into $\mathbb{R}^m$.

It follows from Lemma 4.7.7 that this definition is equivalent to the definition of a system node given in 4.7.2.
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(ii) if we split $S$ into $S = \begin{bmatrix} S_X & S_Y \end{bmatrix}$ in accordance with the splitting of the range space $\begin{bmatrix} X & Y \end{bmatrix}$ ($S_X$ is the ‘top row’ of $S$ and $S_Y$ is the ‘bottom row’), then $S_X$ is closed (with $D(S_X) = D(S)$);

(iii) the operator $A$ defined by $Ax = S_X \begin{bmatrix} x \end{bmatrix}$ with domain $D(A) = \{ x \in X \mid \begin{bmatrix} x \end{bmatrix} \in D(S) \}$ is the generator of a strongly continuous semigroup on $X$;

(iv) for every $u \in U$ there is some $x \in X$ such that $\begin{bmatrix} x \end{bmatrix} \in D(S)$.

It turns out that when these additional conditions hold, then (1.1.2) has trajectories of the following type. We use the operators $S_X$ and $S_Y$ defined in (ii) to split (1.1.2) into

\[
\begin{aligned}
\dot{x}(t) &= S_X \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \\
y(t) &= S_Y \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},
\end{aligned} \tag{1.1.3}
\]

If (i)-(iv) hold, then for each $x_0 \in X$ and $u \in C^2([0, \infty); U)$ such that $\begin{bmatrix} x_0 \\ u(0) \end{bmatrix} \in D(S)$, there is a unique function $x \in C^1([0, \infty); X)$ (called a state trajectory) satisfying $x(0) = x_0$, $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in D(S)$, $t \geq 0$, and $\dot{x}(t) = S_X \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$, $t \geq 0$. If we define the output $y \in C([0, \infty); Y)$ by $y(t) = S_Y \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$, $t \geq 0$, then the three functions $u$, $x$, and $y$ satisfy (1.1.2) (this result is a slightly simplified version of Lemma 4.7.8).

Another consequence of conditions (i)-(iv) above is that it is almost (but not quite) possible to split a system node $S$ into $S = \begin{bmatrix} \Delta & \Gamma \\ \Theta & \Psi \end{bmatrix}$ as in the finite-dimensional case. If $X$ is finite-dimensional, then the operator $A$ in (iii) will be bounded, and this forces the full system node $S$ to be bounded, with $D(S) = \begin{bmatrix} \Delta & \Theta \\ \Gamma & \Psi \end{bmatrix}$. Trivially, in this case $S_X$ can be decomposed into four bounded operators $S = \begin{bmatrix} \Delta & \Theta \\ \Gamma & \Psi \end{bmatrix}$. If $X$ is infinite-dimensional, then a partial decomposition still exists. The operator $A$ in this partial decomposition corresponds to an extension $A_X$ of the semigroup generator $A$ in (iii).\footnote{We shall also refer to $A$ as the main operator of the system node.}

This extension is defined on all of $X$, and it maps $X$ into a larger ‘extrapolation space’ $X_{-1}$ which contains $X$ as a dense subspace. There is also a control operator $B$ which maps $U$ into $X_{-1}$, and the operator $S_X$ defined in (ii) (the ‘top row’ of $X$) is the restriction to $D(S)$ of the operator $[A_X]B$ which maps $\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$ into $X_{-1}$. (Furthermore, $D(S) = \{ \begin{bmatrix} x \end{bmatrix} \in X \mid [A_X]B[\begin{bmatrix} x \end{bmatrix}] \in X \}$. Thus, $S_X$ always has a decomposition (after an appropriate extension of its domain and also an extension of the range space). The ‘bottom row’ $S_Y$ is more problematic, due to the fact that it is not always possible to embed $Y$ as a dense subspace in some larger space $Y_{-1}$ (for example, $Y$ may be finite-dimensional). It is still true,
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however, that it is possible to define an observation operator $C$ with domain $\mathcal{D}(C) = \mathcal{D}(A)$ by $Cx = S_x \left[ \frac{1}{\lambda} \right]$, $x \in \mathcal{D}(A)$. The feedthrough operator $D$ in the finite-dimensional decomposition $A = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$ need not always exist, and it need not be unique. However, this lack of a unique well-defined feedthrough operator is largely compensated by the fact that every system node has a transfer function, defined on the resolvent set of the operator $A$ in (iii). See Section 4.7 for details.\(^3\)

The other main setting that we shall use (and after which this book has been named) is the $L^p$-well-posed setting with $1 \leq p < \infty$. This setting can be introduced in two different ways. One way is to first introduce a system node of the type described above, and then add the requirement that for all $t > 0$, the final state $x(t)$ and the restriction of $y$ to the interval $[0, t)$ depend continuously on $x_0$ and the restriction of $u$ to $[0, t)$. This added requirement will give us an $L^p$-well-posed linear system if we use the $X$-norm for $x_0$ and $x(t)$, the norm in $L^p([0, t); U)$ for $u$, and the norm in $L^p([0, t); Y)$ for $y$.\(^4\) (See Theorem 4.7.13 for details.)

However, it is also possible to proceed in a different way (as we do in Chapter 2) and to introduce the notion of an $L^p$-well-posed linear system without any reference to a system node. In this approach we look directly at the mapping from the initial state $x_0$ and the input function (restricted to the interval $[0, t)$) to the final state $x(t)$ and the output function (also restricted to the interval $[0, t)$). Assuming the same type of continuous dependence as we did above, the relationship between these four objects can be written in the form (we denote the restrictions of $u$ and $y$ to some interval $[s, t)$ by $\pi_{[s,t]} u$, respectively $\pi_{[s,t]} y$)

$$\begin{bmatrix} x(t) \\ \pi_{[0,t]} y \end{bmatrix} = \begin{bmatrix} \mathcal{D}_0^t & \mathcal{M}_0^t \\ \mathcal{C}_0^s & \mathcal{D}_0^s \end{bmatrix} \begin{bmatrix} x_0 \\ \pi_{[0,s]} u \end{bmatrix}, \quad t \geq 0, \quad s \leq t,$$

for some families of bounded linear operator $\mathcal{M}_0^t : X \to X$, $\mathcal{D}_0^t : X \to L^p([0, t); U)$, and $\mathcal{D}_0^s : L^p([0, s); U) \to L^p([0, t); Y)$. If these families correspond to the trajectories of some system node (as described earlier), then they necessarily satisfy some algebraic conditions, with can be stated without any reference to the system node itself. Maybe the simplest way to list these algebraic conditions is to look at a slightly extended version of (1.1.2).

\(^3\) Another common way of constructing a system node is the following. Take any semigroup generator $A$ in $X$, and extend it to an operator $A_{\lambda} x \in \mathcal{D}(X_{\lambda})$. Let $B \in \mathcal{B}(U; X_{\lambda})$ and $\mathcal{C} \in \mathcal{B}(X_{\lambda}; U)$ be arbitrary, where $X_{\lambda}$ is $\mathcal{D}(A)$ with the graph norm. Finally, fix the value of the transfer function to be a given operator in $\mathcal{B}(U; Y)$ at some arbitrary point in $\mathcal{D}(A)$, and use Lemma 4.7.6 to construct the corresponding system node.

\(^4\) Here we could just as well have replaced the interval $[0, t)$ by $(0, t]$ or $[0, \infty)$. However, we shall later consider functions which are defined pointwise everywhere (as opposed to almost everywhere), and then it is most convenient to use half-open intervals of the type $[s, t)$, $s < t$. 

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where the initial time zero has been replaced by a general initial time $s$, namely

$$
\begin{bmatrix}
\dot{x}(t) \\
y(t)
\end{bmatrix} = S
\begin{bmatrix}
x(t) \\
u(t)
\end{bmatrix}, \quad t \geq s, \quad x(s) = x_s,
$$

and to also look at the corresponding maps from $x_s$ and $\pi_{(s,t)}u$ to $x(t)$ and $\pi_{(s,t)}y$ which we denote by

$$
\begin{bmatrix}
x(t) \\
\pi_{(s,t)}y
\end{bmatrix} = \begin{bmatrix}
\mathcal{X}_t & \mathcal{Y}_t \\
\mathcal{E}_t & \mathcal{D}_t
\end{bmatrix}
\begin{bmatrix}
x_s \\
\pi_{(s,t)}u
\end{bmatrix}, \quad s \leq t.
$$

These two-parameter families of bounded linear operators $\mathcal{X}_t$, $\mathcal{Y}_t$, $\mathcal{E}_t$, and $\mathcal{D}_t$ have the properties listed below. In this list of properties we denote the left-shift operator by

$$(\tau^s u)(s) = u(t + s), \quad -\infty < s, t < \infty,$$

and the identity operator by 1.

**Algebraic conditions 1.1.2** The operator families $\mathcal{X}_t$, $\mathcal{Y}_t$, $\mathcal{E}_t$, and $\mathcal{D}_t$ satisfy the following conditions:\(^5\)

(i) For all $t \in \mathbb{R}$,

$$
\begin{bmatrix}
\mathcal{X}_t & \mathcal{Y}_t \\
\mathcal{E}_t & \mathcal{D}_t
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}.
$$

(ii) For all $s \leq t$,

$$
\begin{bmatrix}
\mathcal{X}_s & \mathcal{Y}_s \\
\mathcal{E}_s & \mathcal{D}_s
\end{bmatrix} = \begin{bmatrix}
\mathcal{X}_t & \mathcal{Y}_t \\
\mathcal{E}_t & \mathcal{D}_t
\end{bmatrix}
\begin{bmatrix}
\pi_{(s,t)}u \\
\pi_{(s,t)}y
\end{bmatrix}.
$$

(iii) For all $s \leq t$ and $h \in \mathbb{R}$,

$$
\begin{bmatrix}
\mathcal{X}_{s+h} & \mathcal{Y}_{s+h} \\
\mathcal{E}_{s+h} & \mathcal{D}_{s+h}
\end{bmatrix} = \begin{bmatrix}
\mathcal{X}_s & \mathcal{Y}_s \\
\mathcal{E}_s & \mathcal{D}_s
\end{bmatrix}
\begin{bmatrix}
\pi_{(s,t)}u \\
\pi_{(s,t)}y
\end{bmatrix}.
$$

(iv) For all $s \leq r \leq t$,

$$
\begin{bmatrix}
\mathcal{X}_r & \mathcal{Y}_r \\
\mathcal{E}_r & \mathcal{D}_r
\end{bmatrix} = \begin{bmatrix}
\mathcal{X}_s & \mathcal{Y}_s \\
\mathcal{E}_s & \mathcal{D}_s
\end{bmatrix}
\begin{bmatrix}
\mathcal{X}_r + \mathcal{Y}_r \\
\mathcal{E}_r + \mathcal{D}_r
\end{bmatrix}.
$$

All of these conditions have natural interpretations (see Sections 2.1 and 2.2 for details): (i) is an initial condition, (ii) says that the system is causal, (iii)

\(^5\) By Theorem 2.2.14, these algebraic conditions are equivalent to those listed in Definition 2.2.1.
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says that the system is time-invariant, and (iv) gives a formula for how to patch two solutions together, the first of which is defined on \([s, r]\) and the second on \([r, t]\), and with the initial state of the second solution equal to the final state of the first solution at the ‘switching time’ \(r\). For example, if we take a closer look at the family \(A_t^s\), then (iii) says that \(A_t^s = A_{t-r}^{s-r}\) for all \(s \leq t\), (i) says that \(A_t^0 = 1\), and (iv) says that \(A_t^r A_{t-r}^{s-r}\) for all \(0 \leq r \leq t\). This means that the family \(A_t^s\) is simply a semigroup (it is the semigroup generated by the operator \(A\) of the corresponding system node).

Not only are the conditions (i)–(iv) above necessary for the family \([A_t^s B_t^s C_t^s D_t^s]\) to be generated by a system node \(\mathcal{S}\) through the equation (1.1.4), but they are sufficient as well (when combined with the appropriate continuity assumptions). This will be shown in Chapters 3 and 4 (out of which the former deals exclusively with semigroups). However, it is possible to develop a fairly rich theory by simply appealing to the algebraic conditions (i)–(iv) above (and appropriate continuity conditions), without any reference to the corresponding system node. Among other things, every \(L^p\)-well-posed linear system has a finite growth bound, identical to the growth bound of its semigroup \(A_t^0\). See Chapter 2 for details.

Most of the remainder of the book deals with extensions of various notions known from the theory of finite-dimensional systems to the setting of \(L^p\)-well-posed linear systems, and even to systems generated by arbitrary system nodes. Some of the extensions are straightforward, others are more complicated, and some finite-dimensional results are simply not true in an infinite-dimensional setting. Conversely, many of the infinite-dimensional results that we present do not have any finite-dimensional counterparts, in the sense that these statements become trivial if the state space is finite-dimensional. In many places the case \(p = \infty\) is treated in a slightly different way from the case \(p < \infty\), and the class of \(L^\infty\)-well-posed linear systems is often replaced by another class of systems, the \(Reg\)-well-posed class, which allows functions to be evaluated everywhere (recall that functions in \(L^\infty\) are defined only almost everywhere), and which restricts the set of permitted discontinuities to jump discontinuities.

The last three chapters have a slightly different flavor from the others. We replace the general Banach space setting which has been used up to now by a standard Hilbert space setting, and explore some connections between well-posed linear systems, Fourier analysis, and operator theory. In particular, in Section 10.3 we establish the standard connection between the class of bounded time-invariant causal operators on \(L^2\) and the set of bounded analytic functions on the right half-plane, and in Sections 10.5–10.7 the admissibility and boundedness of scalar control and observation operators for contraction semigroups are characterized by means of the Carleson measure theorem. Chapter 11 has
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a distinct operator theory flavor. It contains among others a systems theory
interpretation of the basic dilation and model theory for continuous-time con-
traction semigroups on Hilbert spaces.

Chapter 12 contains a short introduction to discrete-time systems (and it
also contains a section on continuous-time systems). Some auxiliary results
have been collected in the appendix.

After this rough description of what this book is all about, let us also tell the
reader what this book is not about, and give some indications of where to look
for these missing results.

There are a number of examples of \( L^p \)-well-posed linear systems given in
this book, but these are primarily of a mathematical nature, and they are not the
true physical examples given in terms of partial differential equations which
are found in books on mathematical physics. There are two reasons for this
lack of physical examples. One of them is the lack of space and time. The
present book is quite large, and any addition of such examples would require
a significant amount of additional space. It would also require another year or
two or three to complete the manuscript. The other reason is that the two recent
volumes Lasiecka and Triggiani (2000a, b) contain an excellent collection of
examples of partial differential equations modeling various physical systems.
By Theorem 5.7.3(iii), most of the examples in the first volume dealing with
parabolic problems are \( Reg \)-well-posed. Many of the examples in the second
volume dealing with hyperbolic problems are \( L^2 \)-well-posed. Almost all the
eamples in Lasiecka and Triggiani (2000a, b) are generated by system nodes.
(The emphasis of these two volumes is quite different from the emphasis of
this book. They deal with optimal control, whereas we take a more general
approach, focusing more on input/output properties, transfer functions, coprime
fractions, realizations, passive and conservative systems, discrete time systems,
model theory, etc.)

Our original main motivation for introducing the class of systems generated
by arbitrary system nodes was that this class is a very natural setting for a
study of impedance passive systems. Such systems need not be well-posed,
but under rather weak assumptions they are generated by system nodes. The
decision not to include a formal discussion of impedance passive systems in this
book was not easy. Once more this decision was dictated partly by the lack of
space and time, and partly by the fact that there is another recently discovered
setting which may be even more suitable for this class of systems, namely the
continuous time analogue of the state/signal systems introduced in Arov and
Staffans (2004, see also Ball and Staffans 2003). Impedance passive systems
are discussed in the spirit of this book in Staffans (2002a, b, c).

Another obvious omission (already mentioned above) is the lack of results
concerning quadratic optimal control. This omission may seem even more
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strange in light of the fact that the original motivation for writing this book was to present a general theory that could be used in the study of optimal control problems (of definite and indefinite type). However, also this omission has two valid reasons. The first one is the same as we mentioned above, i.e., lack of space and time. The other reason is even more fundamental: the theory of optimal control is at this very moment subject to very active research, and it has not yet reached the needed maturity to be written down in the form of a monograph. We are here thinking about a general theory in the spirit of this book. There do, of course, exist quite mature theories for various subclasses of systems. One such class is the one which assumes that the system is of the ‘classical’ form (1.1.1), where $A$ is the generator of a strongly continuous semigroup and the operators $B$, $C$, and $D$ are bounded. This class is thoroughly investigated in Curtain and Zwart (1995). Systems of this type are easy to deal with (hence, they have a significant pedagogical value), but they are too limited to cover many of the interesting boundary control systems encountered in mathematical physics. (For example, the models developed in Sections 11.6 and 11.7 have bounded $B$, $C$, and $D$ only in very special cases.) Other more general (hence less complete) theories are found in, e.g., Lions (1971), Curtain and Pritchard (1978), Bensoussan et al. (1992), Fattorini (1999), and Lasiecka and Triggiani (2000a, b). Quadratic optimal control results in the setting of $L^2$-well-posed linear systems are found in Mikkola (2002), Staffans (1997, 1998a, b, c, d), Weiss (2003), and Weiss and Weiss (1997).

There is a significant overlap between some parts of this book and certain books which deal with ‘abstract system theory’, such as Fuhrmann (1981) and Feintuch and Saeks (1982), or with operator theory, such as Lax and Phillips (1967), Sz.-Nagy and Foiaş (1970), Brodskii (1971), Livšic (1973), and Nikol’ski˘ı (1986). In particular, Chapter 11 can be regarded as a natural continuous-time analogue of one of the central parts of Sz.-Nagy and Foiaş (1970, rewritten in the language of $L^2$-well-posed linear systems).

1.2 Overview of chapters 2–13

Chapter 2 In this chapter we develop the basic theory of $L^p$-well-posed linear systems starting from a set of algebraic conditions which is equivalent to 1.1.2. We first simplify the algebraic conditions 1.1.2 by using a part of those conditions to replace the original two-parameter families $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ introduced in Section 1.1 by a semigroup $\mathcal{A}$, $t \geq 0$, and three other operators, the input map $\mathcal{B} = \mathcal{B}^{0,\infty}$, the output map $\mathcal{C} = \mathcal{C}^{0,\infty}$, and the input/output map $\mathcal{D} = \mathcal{D}^{0,\infty}$. The resulting algebraic conditions that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ have to satisfy are listed in 2.1.3 and again in Definition 2.2.1. The connection between the
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The quadruple $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \text{and} \mathfrak{D}$ and the original four operator families $\mathfrak{A}_t, \mathfrak{B}_t, \mathfrak{C}_t, \text{and} \mathfrak{D}_t$ is explained informally in Section 2.1 and more formally in Definition 2.2.6 and Theorem 2.2.14. Thus, we may either interpret an $L^p$-well-posed linear system as a quadruple $\Sigma = \left[ \begin{array} {c|c} A & B \\ \hline C & D \end{array} \right]$, or as a two-parameter family of operators $\Sigma_t = \left[ \begin{array} {c|c} A_t & B_t \\ \hline C_t & D_t \end{array} \right]$, where $s$ represents the initial time and $t$ the final time.

In the case where $p = \infty$ we often require the system to be $\text{Reg}$-well-posed instead of $L^\infty$-well-posed. Here $\text{Reg}$ stands for the class of regulated functions (which is described in more detail in Section A.1). By a regulated function we mean a function which is locally bounded, right-continuous, and which has a left-hand limit at each finite point. The natural norm in this space is the $L^\infty$-norm (i.e., the sup-norm). In this connection we introduce the following terminology (see Definition 2.2.4). By an $L^p|\text{Reg}$-well-posed linear system we mean a system which is either $\text{Reg}$-well-posed or $L^p$-well-posed for some $p$, $1 \leq p \leq \infty$, and by a well-posed linear system we mean a system which is either $\text{Reg}$-well-posed or $L^p$-well-posed for some $p$, $1 \leq p < \infty$. Thus, the $L^p$-case with $p = \infty$ is included in the former class but not in the latter. The reason for this distinction is that not all results that we present are true for $L^\infty$-well-posed systems. Whenever we write $L^p|\text{Reg}$ we mean either $L^p$ or $\text{Reg}$, whichever is appropriate at the moment.

In our original definition of the operators $\mathfrak{B}$ and $\mathfrak{D}$ we restrict their domains to consist of those input functions which are locally in $L^p|\text{Reg}$ with values in $U$, and whose supports are bounded to the left. The original range spaces of $\mathfrak{C}$ and $\mathfrak{D}$ consist of output functions which are locally in $L^p|\text{Reg}$ with values in $Y$. However, as we show in Theorem 2.5.4, every $L^p|\text{Reg}$-well-posed linear system has a finite exponential growth bound (equal to the growth bound of its semigroup). This fact enables us to extend the operators $\mathfrak{B}$ and $\mathfrak{D}$ to a larger domain, and to confine the ranges of $\mathfrak{C}$ and $\mathfrak{D}$ to a smaller space. More precisely, we are able to relax the original requirement that the support of the input function should be bounded to the left, replacing it by the requirement that the input function should belong to some exponentially weighted $L^p|\text{Reg}$-space. We are also able to show that the ranges of $\mathfrak{C}$ and $\mathfrak{D}$ lie in an exponentially weighted $L^p|\text{Reg}$-space (the exponential weight is the same in both cases, and it is related to the growth bound of the system). In later discussions we most of the time use these extended/confined versions of $\mathfrak{B}, \mathfrak{C}, \text{and} \mathfrak{D}$.

As part of the proof of the fact that every $L^p|\text{Reg}$-well-posed linear system has a finite growth bound we show in Section 2.4 that every such system can be interpreted as a discrete-time system $\Sigma = \left[ \begin{array} {c|c} A & B \\ \hline C & D \end{array} \right]$ with infinite-dimensional input and output spaces, and with bounded operators $\Lambda, \mathfrak{B}, \mathfrak{C}, \text{and} \mathfrak{D}$. More precisely, $\left[ \begin{array} {c|c} A & B \\ \hline C & D \end{array} \right] = \left[ \begin{array} {c|c} A_t & B_t \\ \hline C_t & D_t \end{array} \right]$, the discrete-time input space is $L^p|\text{Reg}([0, T); U)$, and the output space is $L^p|\text{Reg}([0, T); Y)$, for some $T > 0$. We achieve
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this by regarding $L^p|\text{Reg}([0, \infty); U)$ as an infinite product of the spaces $L^p|\text{Reg}([0, kT, (k+1)T); U)$, $k = 0, 1, 2, \ldots$, and treating $L^p|\text{Reg}([0, \infty); Y)$ in a similar manner.

In Section 2.6 we show that a linear time-invariant causal operator which maps $L^p|\text{Reg}_{\text{loc}}([0, \infty); U)$ into $L^p|\text{Reg}_{\text{loc}}([0, \infty); U)$ can be interpreted as the input/output map of some $L^p$-well-posed linear system if and only if it is exponentially bounded. In Section 2.7 we show how to re-interpret an $L^p$-well-posed linear system with $p < \infty$ as a strongly continuous semigroup in a suitable (infinite-dimensional) state space. This construction explains the connection between a well-posed linear system and the semigroups occurring in scattering theory studied in, e.g., Lax and Phillips (1967).

Chapter 3 Here we develop the basic theory of $C_0$ (i.e., strongly continuous) semigroups and groups. The treatment resembles the one found in most textbooks on semigroup theory (such as Pazy (1983)), but we put more emphasis on certain aspects of the theory than what is usually done. The generator of a $C_0$ semigroup and its resolvent are introduced in Section 3.2, and the celebrated Hille–Yosida generating theorem is stated and proved in Section 3.4, together with theorems characterizing generators of contraction semigroups. The primary examples are shift semigroups in (exponentially weighted) $L^p$-spaces.

Dual semigroups are studied in Section 3.5, both in the reflexive case and the nonreflexive case (in the latter case the dual semigroup is defined on a closed subspace of the dual of the original state space). Here we also explain the duality concept which we use throughout the whole book: in spite of the fact that most of the time we work in a Banach space instead of a Hilbert space setting, we still use the conjugate-linear dual rather than the standard linear dual (to make the passage from the Banach space to the Hilbert space setting as smooth as possible).

The first slightly nonstandard result in Chapter 3 is the introduction in Section 3.6 of “Sobolev spaces” with positive and negative index induced by a semigroup generator $A$, or more generally, by an unbounded densely defined operator $A$ with a nonempty resolvent set. If we denote the original state space by $X = X_0$, then this is a family of spaces

$$\cdots \subset X_2 \subset X_1 \subset X \subset X_{-1} \subset X_{-2} \subset \cdots,$$

where each embedding is continuous and dense, and $(\alpha - A)$ maps $X_{j+1}$ one-to-one onto $X_j$, for all $\alpha$ in the resolvent set of $A$ and all $j \geq 0$. A similar statement is true for $j < 0$: the only difference is that we first have to extend $A$

In the Russian tradition these spaces are known as spaces with a ‘positive norm’ respectively ‘negative norm’. Spaces with positive index are sometimes referred to as ‘interpolation spaces,’ and those with negative index as ‘extrapolation spaces’.