Introduction

«On donne deux circonférences O et O'. D'un point A pris sur O on mène les tangentes à O'; on joint les points de contact de ces tangentes; on mène la tangente en A à la circonférence O. On demande le lieu du point d'intersection de cette tangente avec la corde des contacts de la circonférence O'.»
Chacun comprenait l'importance d'un pareil théorème.

JULES VERNE
Paris au vingtième siècle

The objective of this book is to give an idea of some modern techniques in the theory of integrable systems, and of how to use them to get topological information, through examples coming from mechanics. I have tried to go straight to the point, explaining the techniques as I go along and delaying the general theory to the appendices.

Our main topic is a "rigid-body-with-a-fixed-point-in-a-constant-gravitational-field", in the cases where the differential equations describing its motion are completely integrable. It is a recent custom – without doubt both unfortunate and convenient – to abbreviate this cumbersome locution to *spinning top* or even *top*. I will follow this.

Spinning tops are celebrated examples of completely integrable systems with two degrees of freedom: a rigid body moving about its centre of mass, a rigid body with an axis of revolution (this is the case that explains – rather badly – the use of the words "spinning top") and lastly the mysterious (?) Kowalevski case.

Much work has been done during the two last decades on integrable systems in both finite and infinite dimensions and a lot of sophisticated techniques have been developed, so that these systems henceforth stand at a crossroads of numerous avenues of mathematics. I will only mention here representation theory (of Lie algebras, loop algebras, Kac-Moody algebras ...) and algebraic geometry (algebraic curves, Abelian varieties, ϑ-functions ...).

In the meantime, but quite separately, some studies of the topology of these systems (Liouville tori and their bifurcations) have appeared.

My starting point was the idea that the "sophisticated techniques" I mentioned might be powerful enough to give some information on the topology as well. This is actually the case and this is what I aim to show in this text. Here the originality is primarily in the approach, in the common method used to handle the different cases. However, as this method is, after all, rather natural, it eventually gives new results about old problems ...

Spinning tops have been investigated since the eighteenth century and it is well known that the solutions of the differential equations can be expressed in terms of elliptic functions or, in the Kowalevski case, of Abelian functions related to a hyperelliptic curve.

Thus, a posteriori, it is known that there are algebraic curves in the landscape. Modern techniques bring them to the fore a priori: there is indeed an algebraic curve, we know it from the beginning; it can be a way to express that the system has enough constants of motions, first integrals; and it can be used to write down the solutions. What I will explain here is how it can also be used to describe the topology of the system.

As an introduction, I will try to give now a rough idea of the method, in a rather discursive style – some of the missing details will be developed in the appendices.

1. Completely integrable systems

A completely integrable system is a Hamiltonian system that admits the maximum possible number of first integrals. The easiest way to be more precise is to state a definition in a symplectic manifold, but it turns out that the natural scenery of almost all examples is a Poisson manifold (a very careful exposition of the Poisson case can be found in Vanhaecke's paper [83]). I will recall here some basic definitions and results, and will content myself with giving hints and references for some of the notions and proofs.

1.1. Poisson structures and Hamiltonian systems

A Poisson manifold is a smooth manifold W, the ring of functions $\mathcal{C}^\infty(W)$ of which is endowed with a Lie algebra structure $\{\,,\,\}$ which is a derivation in both its entries. That is, $\{\,,\,\}$ must be skew-symmetric, satisfy the Jacobi identity

$$\{f,\{g,h\}\}+\{g,\{f,h\}\}+\{h,\{f,g\}\}=0$$

and the Leibniz rule

$$\{f,gh\}=\{f,g\}\,h+g\,\{f,h\}\,.$$

The bracket $\{\,,\,\}$ is the *Poisson bracket*. As any function H on W defines a derivation $\{H,\cdot\}$, it defines a vector field X_H, *the Hamiltonian vector field:* by definition

$$X_H\cdot f=\{H,f\}=df\,(X_H)\,.$$

For instance, in the case of spinning tops, W is the vector space $\mathbf{R}^3\times\mathbf{R}^3$, which we shall consider as "the dual of the Lie algebra of the group of Euclidean motions of the 3-space". This seems to be very intricate, but gives it a Poisson-manifold natural structure: the duals of the Lie algebras are the basic Poisson manifolds (see Appendix 1).

The function H (the Hamiltonian) defines a vector field X_H (the Hamiltonian vector field), which in turn defines a differential equation, the *Hamiltonian system* associated with H.

By skew-symmetry, $\{H,H\}=0$ so that $dH(X_H)=0$ and H is constant along the trajectories of X_H. In other words, the solutions of the Hamiltonian system remain in the levels of H. In "Hamiltonian" mechanics, when H is the total energy of a system, H is constant during the motion.

Any function f having that property (remaining constant along the trajectories), that is, such that $df(X_H)=0$ (or $\{H,f\}=0$) is called a first integral. Notice that a Hamiltonian system always has at least one first integral!

For instance, if $f\in\mathcal{C}^\infty(W)$ is such that $\{f,g\}=0$ for any function g on W, it is a first integral for *any* Hamiltonian system on W. Such functions are called *Casimir functions* – or "trivial" first integrals in the physicists' terminology.

1.2. Completely integrable systems on symplectic manifolds

An important subclass of Poisson manifolds is that of *symplectic* manifolds. A symplectic manifold is, roughly speaking, a manifold W endowed with a non-degenerate 2-form ω. Being non-degenerate, ω allows us to associate, with any function H, a vector field X_H defined by

$$\omega(X_H, \cdot) = dH(\cdot).$$

This defines in turn a bracket on $\mathcal{C}^\infty(V)$:

$$\{f,g\} = \omega(X_g, X_f) = dg(X_f)$$

which is obviously skew-symmetric and satisfies the Leibniz rule. An exercise that is both easy and useful is to check that this bracket satisfies the Jacobi identity if and only if the form ω is *closed*, and this is the reason why a symplectic manifold is a manifold endowed with a non-degenerate closed 2-form.

Of course, the non-degeneracy condition forces the dimension of the manifold to be even, and this is enough to prove that there are much more Poisson than symplectic manifolds (there is a canonical non-trivial Poisson structure on, say, $\mathbf{R}^3$; see the examples in Appendix 1). Nevertheless, symplectic manifolds are Poisson manifolds, functions on them define Hamiltonian systems[1] and we come back to first integrals.

DEFINITION. *A Hamiltonian system on a $2n$-dimensional symplectic manifold is completely integrable (or simply integrable) if it has n functionally independent first integrals $H_1, \ldots, H_n$ that pairwise commute (i.e. such that $\{H_i, H_j\} = 0$).*

Remarks. Write $X_i = X_{H_i}$.

- "Functionally independent" means that there exists an open dense subset U in the manifold V such that, for $x \in U$, the differentials $dH_i(x)$ (or, which amounts to the same thing, the vectors $X_i(x)$) are independent.

- Since H_i and H_j commute, $\omega(X_i, X_j) = 0$: at a generic point of V, the X_i will generate an n-dimensional isotropic subspace of the tangent space T_xV. Since $\dim V = 2n$, n is the maximum possible dimension for isotropic subspaces, so that n is also the maximum possible number of (independent) first integrals. For instance, the Hamiltonian of the system belongs to the algebra generated by the functions H_i.

- A classical exercise (see any textbook on symplectic geometry, e.g. that of Libermann & Marle [59]) is to show that $[X_f, X_g] = \pm X_{\{f,g\}}$, in other words that the vector fields X_i commute.

- Last but not least, I do insist on the fact that *any* function in the algebra generated by $H_1, \ldots, H_n$ will give a Hamiltonian system that admits $H_1, \ldots, H_n$ as first integrals and thus will be completely integrable. There is no reason at this point to require that H_1 should be the Hamiltonian of the system under consideration. What is

[1] In the symplectic manifold $\mathbf{R}^{2n}$ with coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ and symplectic form $\sum dp_i \wedge dq_i$, the equations in the Hamiltonian system associated with H are the celebrated "Hamilton equations". Hence all the terminology used here.

intrinsically defined is actually the algebra generated by the functions H_i, but it will be convenient to use a basis. Once a basis is chosen, we may consider it as a *momentum mapping*

$$H = (H_1, \dots, H_n) ; W \longrightarrow \mathbf{R}^n$$

since this is the momentum mapping for the local $\mathbf{R}^n$-action defined by the flows of the functions H_i.

1.3. A few words on integrable systems on Poisson manifolds

A Poisson manifold is foliated by submanifolds on which the Poisson bracket defines a Poisson structure (it should be noticed that the Poisson bracket defines nothing on a submanifold in general), which is associated with a symplectic form: in short, *symplectic leaves*. I do not want to spend much time on the details of the formulation or the proof[2] of this result.

The simplest case is when the symplectic foliation is described by the Casimir functions. This happens quite often, in particular in the examples we have in view. More precisely, this is the case when the generic leaves are the connected components of the regular common levels of Casimir functions. Then, if $2n$ is the maximal dimension[3] of symplectic leaves and if $2n + m$ is the dimension of the Poisson manifold, the maximum number of independent commuting first integrals one can expect is $m + n$ (m "trivial", n "non-trivial"). Moreover, as the Casimirs commute with everything, the non-trivial integrals define commuting flows on the symplectic leaves in this case. The cases we will consider are those involving orbit foliation for the coadjoint action on the dual of a Lie algebra, for Lie algebras in which that property is satisfied. This is the subject of Appendix 1.

In any case, we can restrict ourselves to symplectic leaves, so that we can content ourselves with the "symplectic" definition above.

Remark. The consideration of general Poisson structures, and even of several Poisson structures on the same manifold, is very useful in *constructing* integrable systems (see e.g. the discussion about the AKS theorem in Appendix 2).

1.4. A list of examples

Finite-dimensional integrable systems include the following[4] examples: the Calogero-Moser systems, the Calogero-Sutherland systems, the Calogero systems, the Clebsch rigid body in an ideal fluid, the dimension-n rigid body, the Euler-Arnold rigid body, the Euler equations, the Euler-Poinsot top, the exotic $SO(4)$-top, the free particle on an ellipsoid, the free rigid body, the Garnier system, the Gaudin system, the geodesic flow on a torus, the geodesics on an ellipsoid, the geodesics on a surface of revolution, the geodesics on quadrics, the geodesics on $SO(3)$, the Goldman functions, the Goryachev-Chaplygin top, the harmonic oscillator, the Hénon-Heiles system, the Holt potential, the

[2] I suggest that the interested reader should look for instance at Libermann & Marle [59] and at the references therein.

[3] Of course, the foliation is singular.

[4] I have used in particular Reyman & Semenov-Tian-Shanski's survey [77], Perelomov's book [70], de Dinteville's paper [25] and my lecture notes [14] to compile this list, which is both incomplete and redundant.

Jeffrey-Weitsman system, the Kepler problem, the Kirchhoff rigid body in an ideal fluid, the Kolosoff potential, the Kovalevskaya[5] rigid body, the Kowalevskaya gyroscope, the Kowalevski top, the Lagrange top, the mathematical pendulum, the Moser systems, the motion of a particle in a central field, the motion of a particle in a potential field, the motion of a particle on a sphere in a quadratic potential, the Neumann problem, the nonabelian Toda lattices, the non-periodic Toda lattice, the partially symmetric gyroscope, the pendulum, the periodic Toda lattice, the Ruijsenaars system, the spherical pendulum, the $SO(n)$ top, the Steklov rigid body in an ideal fluid, the symmetric top, the Toda lattice, the two-body problem, the two-dimensional anharmonic oscillator, the two-dimensional oscillator.

2. The Arnold-Liouville theorem

2.1. What the Arnold-Liouville theorem says

Now we are back in a symplectic manifold V and we have an integrable system as in the definition above. Thus the geometric situation is the following: at a point x of the open dense subset where the integrals H_i are independent, we have n independent vectors; they span a Lagrangian[6] subspace of T_xV that is tangent to the common level of $H_1, \dots, H_n$ in which x stands. If there is any reason of thinking that the flows of the vector fields are complete, we may derive an action of the group $\mathbf{R}^n$ (recall that the vector fields do commute), on any regular level, that is locally free (independence again). Now this gives to the regular levels an affine structure in which the flows under consideration are linear. This is what the Arnold-Liouville theorem says.

THEOREM (Arnold [9]). *Let $h = (h_1, \dots, h_n) \in \mathbf{R}^n$ be a regular value of the mapping $(H_1, \dots, H_n) : V \to \mathbf{R}^n$. Let $\mathcal{T}_h$ be the corresponding regular level, so that $\mathcal{T}_h$ is a Lagrangian submanifold. Let x be a point in $\mathcal{T}_h$. If the flows of the vector fields $X_1, \dots, X_n$ starting at x are complete, the connected component of x in $\mathcal{T}_h$ is a homogeneous space of $\mathbf{R}^n$. In particular, it has coordinates $(\varphi_1, \dots, \varphi_n)$ in which the vector field X_i can be written:*

$$X_i = \sum_{j=1}^{n} \omega_j^i(h) \frac{\partial}{\partial \varphi_j}.$$

Remark. This is actually the easy half of the Arnold-Liouville theorem. The most economical way to ensure that the flows are complete is to require that the component of x is compact (this will be so if one of the integrals is proper). Of course, in this case, it must be a torus (one of the celebrated Liouville tori) and $(\varphi_1, \dots, \varphi_n)$ can be considered as $\bmod 2\pi$ coordinates. This is why they are called *angle* coordinates. Now the Arnold-Liouville theorem states that there are some complementary coordinates, *action*[7] coordinates: they are related to another affine structure, transversal to the levels. I will not discuss action coordinates here, despite their great significance.

[5] As a general rule, I have tried to use the authors' names as they appear in their papers published in French or English, e.g. Sophie Kowalevski.

[6] i.e. isotropic for the symplectic form, and of maximal dimension.

[7] The existence of the action-angle coordinates may be understood as a precise way of saying that the differential system is "integrable by quadratures".

2.2. What the Arnold-Liouville theorem does not say

From the topological viewpoint, the theory seems to be rather poor: there are only tori, and moreover the solutions of the differential systems are (images of) straight lines. Nevertheless some non-trivial questions remain.

1. How is it possible to decide, conceptually (i.e. without too much computation) that an actual level h is regular? In the simple case of spinning tops, as we shall see, the discussion is about a common level of four polynomials in six variables ....

2. How will we know whether the flows are complete?

3. And even is a level is known to be compact, of how many Liouville tori does it consist?

4. What happens if one goes through a critical value?

5. It is quite tautological to claim that the flows are linear with respect to an affine structure that they have themselves defined. Is there any linearisation statement with respect to a more canonical affine structure?

I will now sketch the method that I want to discuss and illustrate in this text. Its main feature is that it provides a framework in which there are natural strategies for investigating and even for answering all the above questions. The idea is to model the levels together with their non-explicit affine structure by objects that are rigid enough to be endowed with a *canonical* affine structure. These are the Abelian varieties. This is no surprise: even if we did not notice them, they appeared in the landscape when I mentioned elliptic functions and/or Abelian integrals.

3. A discourse on the method

Forget integrable systems for the moment. The method will apply to *Lax equations*, that is, differential equations of the form

$$\frac{d}{dt}A = [A, B]$$

where A and B are real or complex matrices depending on time. The bracket is the usual Lie bracket of matrices, so that such an equation expresses at the infinitesimal level the fact that the matrix A remains in the same conjugacy class, in other words that the solutions have the form

$$A(t) = U(t)A(0)U(t)^{-1}$$

for some (unknown!) invertible matrix $U(t)$.

Remark. Notice that, in general, the unknown functions that are the entries of A also appear in the entries of B. In other words, B will be a function of A: despite the notation, the differential equation is nonlinear.

The method I am discussing will actually apply to Lax equations in which A and B have entries in the ring of real or complex Laurent polynomials in a variable λ, which will be called the *spectral parameter*. I prefer to put the dependence on λ into the notation, so that A and B will be called A_λ and B_λ and the equation will have the form

$$\frac{d}{dt}A_\lambda = [A_\lambda, B_\lambda],$$

where the bracket is again that of matrices: $[A_\lambda, B_\lambda] = A_\lambda B_\lambda - B_\lambda A_\lambda$.

For instance, the differential equation

$$\frac{d}{dt}\begin{pmatrix} b_1 & 1 & x_3\lambda \\ x_1 & b_2 & 1 \\ \lambda^{-1} & x_2 & b_3 \end{pmatrix} = \left[\begin{pmatrix} b_1 & 1 & x_3\lambda \\ x_1 & b_2 & 1 \\ \lambda^{-1} & x_2 & b_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & x_3\lambda \\ x_1 & 0 & 0 \\ 0 & x_2 & 0 \end{pmatrix}\right]$$

is a system of twenty-seven equations (coefficients of λ^{-1}, 1, λ in each entry) in six unknown functions, which turns out to be equivalent to

$$\left\{\begin{array}{rcl} \dot{x_1} & = & x_1(b_2 - b_1) \\ \dot{x_2} & = & x_2(b_3 - b_2) \\ \dot{x_3} & = & x_3(b_1 - b_3) \end{array}\right. \qquad \left\{\begin{array}{rcl} \dot{b_1} & = & x_1 - x_3 \\ \dot{b_2} & = & x_2 - x_1 \\ \dot{b_3} & = & x_3 - x_2 \end{array}\right.$$

3.1. The algebro-geometric zoo

One has the right to ask the reasons for doing something so intricate (having added the spectral parameter for instance), and also how to put a given differential equation into this form. I will not discuss the last question here but, rather, I will explain what the Lax form and the spectral parameter give us.

A considerable number of first integrals. Any Lax equation has, a priori, many first integrals: since the matrix A stays in the same conjugacy class, its eigenvalues will be "constants of motion". In other words, the coefficients of the characteristic polynomial of A are first integrals.

A curve. Now, if there is a spectral parameter in the matrix A, the characteristic polynomial is a polynomial in *two* variables

$$P(\lambda, \mu) = \det(A_\lambda - \mu \,\mathrm{Id}).$$

But what is a polynomial in two variables if not the equation of an algebraic curve? The complex curve C of equation

$$P(\lambda, \mu) = 0$$

will be called the *spectral curve:* it describes the eigenvalues, the *spectrum* of the matrices A_λ. The coefficients in the equation of C are the first integrals mentioned above, so that there are actually *many* spectral curves: one for each value of the set of integrals (in other words, the equation $P(\lambda, \mu) = 0$ describes a family of curves). One could or should index the curve by the value to which it corresponds; once we have noticed that a spectral curve corresponds to a common level of the first integrals, we can also index a level by a curve: I will sometimes write $\mathcal{T}_C$ for the level (in the set of matrices) corresponding to the curve C.

The leading figure. Fix a curve C in the family and suppose that, for some value λ of the spectral parameter, A_λ has a simple spectrum. For any eigenvalue μ of A_λ (that is to say, any μ such that $(\lambda, \mu) \in C$) we now have a line in the space $\mathbf{C}^N$ on which our matrices act: the one-dimensional eigenspace of A_λ with respect to μ. Let us now allow λ to vary. Putting all these lines together, what we get is a complex line bundle on the curve C, the *eigenvector bundle*, the fibre of which at a point (λ, μ) is the eigenspace[8] of A_λ with respect to μ.

Now, when A_λ varies in the common level $\mathcal{T}_C$, we can consider all these line bundles together as a mapping

$$\varphi = \varphi_C : \mathcal{T}_C \longrightarrow \operatorname{Pic}(C)$$

with values in the Picard group of (algebraic) complex line bundles over C (see Appendix 4). This map, the *eigenvector mapping*, will be both our main tool and the leading figure in this text.

Remark. Recall now that any component of the Picard group is a complex torus, so that it shows up in the scenery with its own affine structure (which, incidentally, is canonical). One can dream of reading possible angle variables as the linearisation of flows in $\operatorname{Pic}(C)$. More precisely, if $t \mapsto A_\lambda(t)$ is a solution of the Lax equation, let us consider the image curves $t \mapsto \varphi(A_\lambda(t))$. To state that they are straight lines, with linear parametrisation, would not be tautological, the affine structure underlying the statement being defined without any reference to the vector field in question.

I must also emphasise that the first integrals, the curve and the eigenvector mapping itself do not depend on the matrix B_λ. This is no surprise: in the case of an integrable system $(H_1, \ldots, H_n)$, to each function in the algebra generated by $(H_1, \ldots, H_n)$ will correspond a different B_λ. Our algebraic data describe the levels as a whole, not a special flow on them.

3.2. The case of an integrable system

All this discussion applies very readily to any Lax equation, but we want to apply it to an integrable system. Of course, though Lax equations always have a lot of first integrals, nothing forces these to commute (we do not even have a Poisson structure to give a meaning to this), and, of course, there might not be enough of them. Actually, there is a machinery that allows us to construct Lax equations that *are* integrable systems: this is the so-called Adler-Kostant-Symes theorem (see Appendix 2). For the moment, I will just *assume* that the Lax equation under consideration is indeed an integrable system. Also, to simplify the discussion, I will often assume that the flows are complete. In this case, the level $\mathcal{T}_C$ is a homogeneous[9] space of some $\mathbf{R}^n$, which is, moreover, described by polynomial equations. The complexified level $\mathcal{T}_C^{\mathbf{C}}$ (the complex solutions of the same polynomial equations) could thus perfectly well be an open subset of a complex torus.

[8] There always exist special values of λ for which the spectrum of A_λ is not simple. However the complex line bundle is well defined, at least if the curve C is smooth (see Appendix 3).

[9] I am implicitly assuming that the matrices A_λ and B_λ are real matrices, i.e. have real entries, although I make them act on the complex vector space $\mathbf{C}^N$.

To dream. Can we use the eigenvector mapping φ_C to answer all the questions raised in 2.2? If the differential system were real, the mapping φ_C would preserve all real structures and send $\mathcal{T}_C$ (the real part of $\mathcal{T}_C^{\mathbf{C}}$) into the real part of $\operatorname{Pic}(C)$, which is quite easy to describe.

To wake up. The eigenvector mapping φ_C can almost never be an isomorphism.

1. Dimensions may not fit. First of all, there is no reason why $\mathcal{T}_C^{\mathbf{C}}$ and $\operatorname{Pic}(C)$ should have the same dimension. In the case of the rigid body, $\dim \mathcal{T}_C = 2$, but we shall get Picard groups of dimension 1 for the symmetric (Lagrange) top and 5 for the Kowalevski top. Notice that $\dim \mathcal{T}_C$ is a datum that comes with the integrable system: it is half the dimension of the symplectic (phase) manifold – what physicists call the number of degrees of freedom. On the contrary, $\dim \operatorname{Pic}(C)$ is the genus of the spectral curve (see Appendix 4) and is related to both the degree in λ of A_λ and the size N of the matrices: it depends on the Lax form we have used. However, it can happen that the dimensions are the same: it is the case for systems such as geodesics on an ellipsoid, the Neumann problem[10] and the periodic Toda lattice, as also for the "dimension-3 free rigid body", a one-degree-of-freedom system, the first to be investigated in this book.

2. The compactness problem. The mapping φ_C cannot be a complex isomorphism, simply because $\mathcal{T}_C^{\mathbf{C}}$ is defined by polynomial equations in a linear space and is an *affine* algebraic variety while the components of $\operatorname{Pic}(C)$ are (compact) tori. Even if the dimensions fitted, φ_C could, at best, send $\mathcal{T}_C^{\mathbf{C}}$ onto an open subset of $\operatorname{Pic}(C)$.

3. The real part. Let us assume that one of the first integrals is proper, so that the real level $\mathcal{T}_C$ is compact. In general, the real part of $\operatorname{Pic}(C)$ will intersect the above mentioned hypersurfaces, so that even the restriction of φ_C to the real part cannot be an isomorphism onto its image.

What happens quite often is that $\dim \mathcal{T}_C \leq \dim \operatorname{Pic}(C)$ and φ_C is a finite covering of an open subset in an Abelian subvariety of $\operatorname{Pic}(C)$. A very common situation for two degrees of freedom is that of a genus-3 spectral curve endowed with an involution τ with four fixed points and where, for some reason, it is known that φ_C takes its values in the "anti-fixed" points of τ, the Abelian variety $\operatorname{Prym}(\tau)$ (see Appendix 5).

3.3. What can nevertheless be done

Regular levels and linearisation. Questions 1 and 5 that I have raised in 2.2 have natural answers in this framework. It is possible to discuss in great generality (that is, for a Lax equation, with no additional integrability assumption) whether φ_C linearises the flow. In almost all the cases we will need to consider, the matrix B_λ occurring in the Lax equation will have a very specific form, so that the eigenvector mapping will automatically linearise the solutions of the Lax equation (see the work of Griffiths [36] and Reyman [74] as well as Appendix 3). These linearisation theorems are simple consequences of determining the tangent mapping of φ_C, according to which it is very often possible to prove statements like "if C is smooth, the corresponding level is regular". Notice that nothing prevents the spectral curve from being singular for all values of the first integrals (a very natural example of this situation will be explained in III.3.2).

[10] See the papers of Moser [63], Knörrer [52] and the author [11, 13].

Liouville tori and their bifurcations. Assume now that one of the first integrals is proper (this is the case for the total energy in the case of tops), so that the levels are compact and the flows complete. Look at Questions 3 and 4 of 2.2. Of course, the fact that φ_C is a covering of its image forbids its direct use to enumerate the Liouville tori. However, it is often possible to modify it, or to understand it well enough to use it to identify the levels with the real part of an Abelian variety – but one has to find another trick for any other case. I have already mentioned that the real part of a Jacobian is easy to investigate once the real structure of the curve itself is well understood (see Appendix 4). The case of a Prym is somewhat more difficult but is still quite feasible.

Varying the values of the first integrals, one gets *families* of Abelian varieties, which may be used to study the critical levels and the bifurcations of the Liouville tori.

Non-compactness. In the general case of a Lax equation, the levels $\mathcal{T}_C$ are common levels of a family of polynomials defined on a linear space, so that the complexified levels are never compact, as I have already mentioned. The image of the eigenvector mapping in the (complex, compact) Jacobian is in general well understood: very often, the complement is related to the Θ-divisor of the curve (see Appendix 4). To say that the (real) levels are non-compact amounts to saying that their image meets the "divisor at infinity": we thus still have the technology to describe the topology in this case (see Chapter V).

3.4. Another approach

There is another possible approach, which turns out to be dual to the one discussed so far: one can consider directly a (complex) level set as an affine algebraic variety, "simply" by adding a divisor at infinity and proving that the completed variety is an Abelian variety, on which the Hamiltonian vector field can be extended in a constant (that is, linear) vector field. In general, the method used to obtain this divisor at infinity is to look for all the Laurent series (in time) that are formal solutions of the system, just by substitution; then the divisor is the locus of the poles. This method relies on algebraic computations that are often tedious. Moreover, the determination of these points at infinity uses a specific vector field to depict the common level sets of several functions, which I find quite unsatisfying. In its favour, the method shows the level set directly in an Abelian variety, avoiding the problem of coverings (there are many papers on this subject, e.g. those of Haine [39], Adler & van Moerbeke [5, 6] and Vanhaecke [82]).

3.5. And if there is no spectral parameter?

All the constructions above make an essential use of the spectral parameter λ: the latter is responsible for the existence of the spectral curve and hence of all the algebro-geometric zoo presented here. There are honest (i.e. natural) integrable systems that have a Lax form without spectral parameter. There are at least two possible reasons for this:

- either, it is known a priori that no curve should be present, for instance because the solutions are known to be exponentials (and not Abelian functions), which is for example the case for the non-periodic Toda lattices, as shown by Flaschka & Haine in [30] (see also Flaschka's lectures in [29]),
- or, the Lax form of the equation cannot be used to solve the equations and/or study the topology. This is the case e.g. for the Euler equations (see Chapter IV) and for