I. Mechanics of Particle Collisions

The dynamics of a granular gas is completely determined by pure mechanical collisions of its particles. Therefore, any theoretical description of a granular gas requires detailed understanding of the collision of two isolated grains.

The only characteristics of pairwise particle collisions are the particles’ masses and radii and their mechanism of dissipation of mechanical energy during collisions, expressed by a single number – the coefficient of restitution.

In this part we discuss in detail the coefficient of restitution as a function of material properties, particle size and impact velocity. We analyse the conservative and dissipative parts of the interaction force of colliding viscoelastic spheres and derive the coefficient of restitution by integrating Newton’s equation of motion.

To illustrate the importance of the correct description of particle collisions for the overall behaviour of granular systems, we analyse the dynamics of two simple few-particle systems. It is shown that the simplifying assumption of a constant coefficient of restitution may lead to an inadequate description of granular systems.
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PARTICLE COLLISIONS

We introduce the notation to describe the collision of particles by using the coefficient of restitution and raise the question whether the coefficient of restitution is adequate enough to describe the collisions of particles in a granular gas. We look at the motion of two colliding spheres of mass $m_1$ and $m_2$ whose velocities are termed $\vec{v}_1$ and $\vec{v}_2$ before the collision and $\vec{v}'_1$ and $\vec{v}'_2$ after the collision.

2.1 Collision on a line

First, we look at the one-dimensional problem, that is, two particles suffering a collision when moving along a line. Before the collision their relative velocity is $v_{12} \equiv v_1 - v_2 < 0$. Due to conservation of the total momentum, the collision cannot change the centre of the mass motion, but can alter the relative velocity. Elastic particles will suffer a rebound with $v'_{12} = -v_{12}$, conserving the kinetic energy of the relative motion. At a dissipative collision, however, part of the energy of the relative motion is lost. Therefore, after the contact the relative velocity will only partly be restored. The coefficient of restitution quantifies this phenomenon:

$$\varepsilon \equiv -\frac{v'_{12}}{v_{12}}. \quad (2.1)$$

According to the dissipative nature of particle collisions the kinetic energy of the relative motion decreases and, hence, $0 \leq \varepsilon \leq 1$. For $\varepsilon = 1$ the total kinetic energy is conserved, that is, the collision occurs elastically. If $\varepsilon = 0$, the complete energy of the relative motion is lost, that is, after a collision the particles are bound together.

Conservation of the total momentum and (2.1) yield the closed set

$$m_1 v'_1 + m_2 v'_2 = m_1 v_1 + m_2 v_2, \quad v'_1 - v'_2 = -\varepsilon (v_1 - v_2), \quad (2.2)$$

which determines the velocities of the particles after the collision:

$$v'_1 = v_1 - \frac{m_{\text{eff}}}{m_1} (1 + \varepsilon) v_{12}, \quad v'_2 = v_2 + \frac{m_{\text{eff}}}{m_2} (1 + \varepsilon) v_{12} \quad (2.3)$$

Throughout this book we have marked particle properties such as velocities, angular velocities, etc. after a collision with a prime.
Fig. 2.1. The particles approach each other with relative velocity \( \vec{v}_{12} = \vec{v}_1 - \vec{v}_2 \).

The geometry of the collision may range from a head-on collision (a) to a glancing one (b). It is characterized by the unit vector \( \vec{e} \) (c).

with the effective mass
\[
m_{\text{eff}} \equiv \frac{m_1 m_2}{m_1 + m_2}.
\]

For elastic particles (\( \varepsilon = 1 \)) and identical masses (\( m_{\text{eff}}/m_1 = m_{\text{eff}}/m_2 = 1/2 \)) the collision causes exchange of the velocities, \( v'_1 = v_2, \ v'_2 = v_1 \).

We have focused on the simple one-dimensional motion to illustrate the basic concept. The geometry of the impact is trivial in this case. We now turn to the more relevant three-dimensional case.

### 2.2 Collision in space

Similarly to the one-dimensional case, the collective motion of the particles does not influence the collision, that is, we can restrict to consider the relative velocity \( \vec{v}_{12} = \vec{v}_1 - \vec{v}_2 \). The geometry of the collision, however, may be very different from the one-dimensional case. It may range from a head-on collision (Fig. 2.1(a)) to a glancing collision (Fig. 2.1(b)). We specify the geometry of the collision by the unit vector
\[
\vec{e} \equiv \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|},
\]

where \( \vec{r}_{12} = \vec{r}_1 - \vec{r}_2 \) is the vector which joins the centres of the spheres at the very moment when they come into contact. The relative velocity of the particles can be decomposed into the normal part, \( \vec{g} \equiv (\vec{v}_{12} \cdot \vec{e}) \vec{e} = \vec{g}_{12}^n \) and the tangential part, \( \vec{g}_{12}^t \equiv \vec{v}_{12} - \vec{g}_{12}^n \). In this section we only look at the normal part and assume that the tangential velocity is not affected by the collision, that is, \( \vec{g}_{12}^t = 0 \).

More general analysis of the tangential motion at the impact is given in Section 3.4. In normal direction, the three-dimensional collision may be considered as an effective 1d collision which defines the (normal) coefficient of restitution
\[
\vec{g}' = -\varepsilon \vec{g}.
\]

This, together with the velocity of the centre of mass \( m_{\text{eff}} (\vec{v}_1/m_1 + \vec{v}_2/m_2) \), which remains constant, constitute the velocities of the particles after the collision.
2.3 Does the coefficient of restitution suffice to describe granular gas dynamics?

The coefficient of restitution, albeit defined for a collision of an isolated pair of colliding particles, is the central characteristic of a granular gas which reflects its dissipative properties. All effects which give granular gases a special position with regard to molecular gases are consequences of dissipative particle collisions, expressed by the coefficient $\varepsilon < 1$.

Granular gases are complex many-particle systems. Hence, the question arises whether such a simple concept as the coefficient of restitution is adequate enough to characterise the dissipative gas behaviour. In general, the dynamics of a many-particle system is governed by Newton’s equations of motion

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}(\mathbf{r}_1, \ldots, \mathbf{r}_N, \mathbf{v}_1, \ldots, \mathbf{v}_N), \quad i = 1, \ldots, N, \quad (2.8)$$

which is a system of $N$ coupled differential equations.

Granular particles exert forces only if they get in contact, that is, there are no long-range forces. Moreover, in dilute granular gases the typical duration of collisions is negligible as compared to the mean duration of free flight in between successive collisions. Hence, in the absence of external fields such as gravity the particles only feel forces during short periods of contact. In between the collisions they move with constant velocity along straight lines. Consequently, for dilute granular gases the dominating number of contacts occurs between pairs of particles. Contacts between three or more particles occur extremely rarely and may be neglected (see also the discussion in Section 1.4 and Exercise 5.1). Instead of the full $N$-particle problem (2.8) the dynamics may be described by successively occurring two-particle collisions. The mechanical interaction of two macroscopic bodies $i$ and $j$ is described by

$$m_{\text{eff}} \ddot{\mathbf{r}}_{ij}(t) = \mathbf{F}[\mathbf{r}_{ij}(t), \mathbf{v}_{ij}(t)] \quad (2.9)$$

with $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$ and $\mathbf{v}_{ij} \equiv \mathbf{v}_i - \mathbf{v}_j$. Equation (2.9) is in fact a one-body problem of motion of an effective particle of mass $m_{\text{eff}}$ subjected to the force $\mathbf{F}[\mathbf{r}_{ij}(t), \mathbf{v}_{ij}(t)]$.
Only the full solution of (2.9) together with appropriate initial conditions describes the collision strictly. Nevertheless the concept of the coefficient of restitution can be motivated to characterize particle interaction in granular gases: The assumption that the number of three-particle collisions is negligible corresponds to the assumption that particle collisions are instantaneous events. Any particle \( i \) collides with particle \( j \) not before the preceding collision of particle \( i \) with another particle \( k \) has been accomplished. Therefore, the detailed dynamics of a single collision due to (2.9) is irrelevant for the dynamics of a granular gas. Instead, only the result of a collision matters, that is, the transformation of the velocities \( \vec{v}_i \) and \( \vec{v}_j \) immediately before the collision into the after-collisional velocities \( \vec{v}_i' \) and \( \vec{v}_j' \). It is this information that is precisely provided by the coefficient of restitution and the corresponding collision rule (2.7).

Since for an isolated pair of colliding particles the descriptions (2.7) and (2.9) must agree, there is a direct relation between the coefficient of restitution and the interaction force in (2.9). Therefore, the coefficient of restitution can be derived from Newton’s equation of motion which is focused upon in the next section. Obviously, since the motion of colliding bodies depends on material properties, sizes of the particles, initial conditions, etc., the coefficient of restitution which describes the collision consequently depends on these values as well. Thus, the coefficient of restitution of realistic particles does not qualify as a constant. The fact that \( \varepsilon \) depends on the impact velocity is experimentally known for a long time (e.g., Hodkinson, 1835). In Section 3.2.1 we prove that the assumption of a constant coefficient of restitution does not agree with the mechanics of materials.

The assumption of a constant coefficient of restitution \( \varepsilon = \text{const.} \) is very helpful when performing calculations since it simplifies the mathematics significantly. We will show, however, that a physically justifiable coefficient, which is a function of the impact velocity \( \varepsilon = \varepsilon(g) \), leads to many exciting effects.

We wish to mention that situations arise when the assumption of a constant coefficient of restitution is well justified, provided there is a narrow velocity distribution which is, moreover, constant in time. This applies, for example, to dilute shear flow in narrow containers or to small containers with heated walls.

A granular gas, as considered in this book, is an ensemble of purely mechanically interacting particles. This restriction might be decisive for the application of the theory. For Earth-bound systems it is known that grains may be charged (i.e., interact not only by contacts) and for astrophysical systems it is plausible that particles are not electrically neutral either. In the subsequent sections we will see that even with this restriction the kinetic theory of granular gases cannot be qualified as trivial. If it is wished to deal with charged particles, that is, with long-range interactions, large part of the theory has to be revised.

**Exercise 2.3** Assume \( \varepsilon = \text{const.} \) A particle falls from height \( H \) and rebounces recurrently from the floor. At which time \( t_{\infty} \) does it come to rest? Can \( t_{\infty} \) be finite?

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3The term *heated* is meant with respect to the *granular temperature*, see Sec. 5.1.